Quantum walks on graphs: state transfer

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Continuous-time quantum walk

”... quantum walk can be regarded as a universal computational primitive, with any desired quantum computation encoded entirely in some underlying graph.” Andrew Childs arXiv:0806.1972
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Transition matrix

\[ U(t) = \exp(itA) \]
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Continuous-time quantum walk

Transition matrix

\[ U(t) = \exp(itA) \]

\[ = I + itA - \frac{1}{2!} t^2 A^2 - \frac{i}{3!} t^3 A^3 + \cdots \]
Time incrementing by 0.25.
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Perfect state transfer

For $P_3$: 
Perfect state transfer

For $P_3$:

$$U(t) = e^{it\theta} \begin{pmatrix} 0.5 & 0.0 & -0.5 \\ 0.0 & 0.0 & 0.0 \\ -0.5 & 0.0 & 0.5 \end{pmatrix} + e^{it\sqrt{2}} \begin{pmatrix} 0.25 & \sqrt{2} & 0.25 \\ \sqrt{2} & 0.5 & \sqrt{2} \\ 0.25 & \sqrt{2} & 0.25 \end{pmatrix} + e^{-it\sqrt{2}} \begin{pmatrix} 0.25 & -\frac{\sqrt{2}}{4} & 0.25 \\ -\frac{\sqrt{2}}{4} & 0.5 & -\frac{\sqrt{2}}{4} \\ 0.25 & -\frac{\sqrt{2}}{4} & 0.25 \end{pmatrix}$$
Perfect state transfer

For $P_3$:

$$U(t) = e^{i t 0} \begin{pmatrix} 0.5 & 0.0 & -0.5 \\ 0.0 & 0.0 & 0.0 \\ -0.5 & 0.0 & 0.5 \end{pmatrix} + e^{i t \sqrt{2}} \begin{pmatrix} 0.25 & \frac{\sqrt{2}}{4} & 0.25 \\ \frac{\sqrt{2}}{4} & 0.5 & \frac{\sqrt{2}}{4} \\ 0.25 & \frac{\sqrt{2}}{4} & 0.25 \end{pmatrix}$$

$$+ e^{-i t \sqrt{2}} \begin{pmatrix} 0.25 & -\frac{\sqrt{2}}{4} & 0.25 \\ -\frac{\sqrt{2}}{4} & 0.5 & -\frac{\sqrt{2}}{4} \\ 0.25 & -\frac{\sqrt{2}}{4} & 0.25 \end{pmatrix}$$

$$U\left(\frac{\pi}{\sqrt{2}}\right) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$
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$$U(t) = e^{it0} \begin{pmatrix} 0.5 & 0.0 & -0.5 \\ 0.0 & 0.0 & 0.0 \\ -0.5 & 0.0 & 0.5 \end{pmatrix} + e^{it\sqrt{2}} \begin{pmatrix} 0.25 & \frac{\sqrt{2}}{4} & 0.25 \\ \frac{\sqrt{2}}{4} & 0.5 & \frac{\sqrt{2}}{4} \\ 0.25 & \frac{\sqrt{2}}{4} & 0.25 \end{pmatrix}$$

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$$U(\frac{\pi}{\sqrt{2}})e_0 = (-1)e_2$$
Perfect state transfer

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perfect state transfer from $a$ to $b$: there exists $\tau$ such that

$$U(\tau)e_a = \gamma e_b$$

where $|\gamma| = 1$. 
Perfect state transfer: paths

Theorem (Godsil 2012)

The only paths which admit perfect state transfer are $P_2$ and $P_3$. 
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Pretty good state transfer from a to b:
Perfect state transfer: paths

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The only paths which admit perfect state transfer are $P_2$ and $P_3$.

Pretty good state transfer from a to b:
for every $\epsilon > 0$, there exists $\tau$ such that
\[ ||U(\tau)e_a - \gamma e_b|| < \epsilon \]
where $|\gamma| = 1$. 
Pretty good state transfer

Theorem (Godsil, Kirkland, Severini, and Smith 2012)

$P_n$ has pretty good state transfer
if and only if $n + 1$ is a prime, twice a prime or a power of 2.
Pretty good state transfer

Theorem (Godsil, Kirkland, Severini, and Smith 2012)

$P_n$ has pretty good state transfer between the ends if and only if $n + 1$ is a prime, twice a prime or a power of 2.
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$P_n$ has pretty good state transfer between the ends if and only if $n + 1$ is a prime, twice a prime or a power of 2.

Theorem (Couthinho, Guo and van Bommel 2017)

$P_n$ has pretty good state transfer between internal vxs if and only if $n + 1 = 2^r p$ where $p$ is a prime.
Pretty good state transfer
Pretty good state transfer
Pretty good state transfer
Pretty good state transfer
Pretty good state transfer
Pretty good state transfer
NO  Pretty good state transfer
Perfect state transfer on paths revisited
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Conjecture (Casaccino, Lloyd, Mancini, and Severini ‘09)

For any $n$, one can find $\alpha$ so that there is perfect state transfer from $\bullet$ to $\bullet$ in $P_n$. 
Perfect state transfer on paths revisited

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For any $n$, one can find $\alpha$ so that there is \textbf{perfect} state transfer from $\bullet$ to $\bullet$ in $P_n$.

Theorem (Kempton, Lippner and Yau 2016)

This is not possible for any $n > 3$. 
Perfect state transfer on paths revisited

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For any \( n \), one can find \( \alpha \) so that there is perfect state transfer from \( \bullet \) to \( \bullet \) in \( P_n \).

Theorem (Kempton, Lippner and Yau 2016)

This is not possible for any \( n > 3 \).

Theorem (Kempton, Lippner and Yau 2017)

For any \( n \), one can find \( \alpha \) so that there is pretty good state transfer from \( \bullet \) to \( \bullet \) in \( P_n \).
Perfect state transfer in strongly regular graphs

Theorem (Coutinho, Godsil, Guo and Vanhove 2015)

A strongly regular graph admits perfect state transfer if and only if it is the complement of a $mK_2$ with $m$ even.
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A strongly regular graph admits perfect state transfer if and only if it is the complement of a mK2 with m even.
Theorem (Godsil, Guo, Kempton and Lippner 2017+)

For any strongly regular graph coming from an orthogonal array, there exists $\alpha$ and $\beta$ such that the $(\alpha, \beta)$-perturbation admits perfect state transfer.
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For any strongly regular graph coming from an orthogonal array, there exists $\alpha$ and $\beta$ such that the $(\alpha, \beta)$-perturbation admits perfect state transfer.

We determine the characteristic polynomial of the perturbed graph in terms of other characteristic polynomials, walk generating functions and $\alpha$ and $\beta$, for any graph.
Theorem (Godsil, Guo, Kempton and Lippner 2017+)

For any strongly regular graph coming from an orthogonal array, there exists $\alpha$ and $\beta$ such that the $(\alpha, \beta)$-perturbation admits perfect state transfer.

- We determine the characteristic polynomial of the perturbed graph in terms of other characteristic polynomials, walk generating functions and $\alpha$ and $\beta$, for any graph.

- For an SRG, the perturbation has only 5 distinct eigenvalues and we are able to find sufficient conditions for p.s.t.
Theorem (Godsil, Guo, Kempton and Lippner 2017+)

For any strongly regular graph, there exists $\alpha$ and $\beta$ such that the $(\alpha, \beta)$-perturbation admits pretty good state transfer.
Theorem (Godsil, Guo, Kempton and Lippner 2017+)

For any strongly regular graph, there exists $\alpha$ and $\beta$ such that the $(\alpha, \beta)$-perturbation admits pretty good state transfer.

In fact, the ”good” values of $\alpha, \beta$ are dense in the reals.
Open problems

Pretty good state transfer

Given an $\epsilon$ in a graph with pretty good state transfer, when does the "$\epsilon$-close" state transfer occur?
Open problems

Pretty good state transfer

Given an $\epsilon$ in a graph with pretty good state transfer, when does the ”$\epsilon$-close” state transfer occur?

Perfect state transfer

Spectral characterization of graphs with a time $\tau$ such that

uniform mixing at time $\tau$; and

perfect state transfer at time $2\tau$
Open problems

Pretty good state transfer

Given an $\epsilon$ in a graph with pretty good state transfer, when does the "$\epsilon$-close" state transfer occur?

Perfect state transfer

Spectral characterization of graphs with a time $\tau$ such that

uniform mixing at time $\tau$; and

periodicity at time $4\tau$ ($U(4\tau)$ is a diagonal matrix)
Thanks!