On some cycles in linearized Wenger graphs

Ye Wang

Shanghai Lixin University of Accounting and Finance, Shanghai, 201209, China

AEGT, August 2017
Contents

1 Linearized Wenger graphs
   ■ Definition of $L_m(q)$
   ■ Property of $L_m(q)$

2 Main results
   ■ Embedding cycles in $L_1(p)$
   ■ Constructing cycles in $L_1(p)$

3 Future work
Definition of Wenger graphs $W_m(q)$

- Let $q$ be a prime power, and let $\mathbb{F}_q$ be the finite field of $q$ elements. For any integer $m$ with $m \geq 1$, the vertex set $V(W_m(q))$ is the disjoint union of two copies of the $m + 1$ dimensional vector space $\mathbb{F}_q^{m+1}$ over finite field $\mathbb{F}_q$, one denoted by $P_{m+1}$ and the other by $L_{m+1}$.
Definition of Wenger graphs $W_m(q)$

- Let $q$ be a prime power, and let $\mathbb{F}_q$ be the finite field of $q$ elements. For any integer $m$ with $m \geq 1$, the vertex set $V(W_m(q))$ is the disjoint union of two copies of the $m + 1$ dimensional vector space $\mathbb{F}_q^{m+1}$ over finite field $\mathbb{F}_q$, one denoted by $P_{m+1}$ and the other by $L_{m+1}$.

- Elements of $P_{m+1}$ will be called points and those of $L_{m+1}$ lines.
Definition of $W_m(q)$

- Let $q$ be a prime power, and let $\mathbb{F}_q$ be the finite field of $q$ elements. For any integer $m$ with $m \geq 1$, the vertex set $V(W_m(q))$ is the disjoint union of two copies of the $m+1$ dimensional vector space $\mathbb{F}_q^{m+1}$ over finite field $\mathbb{F}_q$, one denoted by $P_{m+1}$ and the other by $L_{m+1}$.

- Elements of $P_{m+1}$ will be called points and those of $L_{m+1}$ lines.

- The point $p = (p(1), p(2), \ldots, p(m+1))$ is adjacent to the line $l = [l(1), l(2), \ldots, l(m+1)]$ if and only if

$$p(i) + l(i) = p(1)(l(1))^{i-1},$$

for $i = 2, 3, \ldots, m+1$. 
Definition of linearized Wenger graphs $L_m(q)$ (Cao et al. (2015))

- Let $q$ be the power of prime $p$, and let $\mathbb{F}_q$ be the finite field of $q$ elements. For any integer $m$ with $m \geq 1$, the vertex set $V(W_m(q))$ is the disjoint union of two copies of the $m + 1$ dimensional vector space $\mathbb{F}_q^{m+1}$ over finite field $\mathbb{F}_q$, one denoted by $P_{m+1}$ and the other by $L_{m+1}$. 
Definition of $L_m(q)$

Definition of linearized Wenger graphs $L_m(q)$ (Cao et al. (2015))

- Let $q$ be the power of prime $p$, and let $\mathbb{F}_q$ be the finite field of $q$ elements. For any integer $m$ with $m \geq 1$, the vertex set $V(W_m(q))$ is the disjoint union of two copies of the $m + 1$ dimensional vector space $\mathbb{F}_q^{m+1}$ over finite field $\mathbb{F}_q$, one denoted by $P_{m+1}$ and the other by $L_{m+1}$.

- Elements of $P_{m+1}$ will be called points and those of $L_{m+1}$ lines.
Definition of linearized Wenger graphs $L_m(q)$ (Cao et al. (2015))

- Let $q$ be the power of prime $p$, and let $\mathbb{F}_q$ be the finite field of $q$ elements. For any integer $m$ with $m \geq 1$, the vertex set $V(W_m(q))$ is the disjoint union of two copies of the $m + 1$ dimensional vector space $\mathbb{F}_q^{m+1}$ over finite field $\mathbb{F}_q$, one denoted by $P_{m+1}$ and the other by $L_{m+1}$.

- Elements of $P_{m+1}$ will be called points and those of $L_{m+1}$ lines.

- The point $p = (p(1), p(2), \ldots, p(m+1))$ is adjacent to the line $l = [l(1), l(2), \ldots, l(m+1)]$ if and only if

  $$p(i) + l(i) = p(1)(l(1))^{p^{i-2}},$$

  for $i = 2, 3, \ldots, m + 1$. 
Property of linearized Wenger graphs $L_m(q)$

- The graphs $L_m(q)$ have $2q^{m+1}$ vertices, $q^{m+2}$ edges and are $q$-regular.
Property of linearized Wenger graphs $L_m(q)$

- The graphs $L_m(q)$ have $2q^{m+1}$ vertices, $q^{m+2}$ edges and are $q$-regular.
- Cao, Lu, Wan, Wang and Wang (2015) determine the girth, diameter and the spectrum of linearized Wenger graphs. For $q = p^e$, their results imply that the graphs $L_e(q)$ are expanders.
Property of linearized Wenger graphs $L_m(q)$

- The graphs $L_m(q)$ have $2q^{m+1}$ vertices, $q^{m+2}$ edges and are $q$-regular.
- Cao, Lu, Wan, Wang and Wang (2015) determine the girth, diameter and the spectrum of linearized Wenger graphs. For $q = p^e$, their results imply that the graphs $L_e(q)$ are expanders.
- A work of Alexander, Lazebnik and Thomason (2016) implies that for a fixed $e$ and large $p$, graphs $L_e(p^e)$ are hamiltonian.
Cycles in Wenger graphs

- For Wenger graphs $W_m(q)$, Shao, He and Shan (2008) showed that for any $m \geq 2$, and any $k$ with $k \neq 5$, $4 \leq k \leq 2p$, $W_m(q)$ contains cycles of length $2k$. 
Cycles in Wenger graphs

- For Wenger graphs $W_m(q)$, Shao, He and Shan (2008) showed that for any $m \geq 2$, and any $k$ with $k \neq 5$, $4 \leq k \leq 2p$, $W_m(q)$ contains cycles of length $2k$.

- Wang, Lazebnik, Thomason (2014) extended their results by showing $W_m(q)$ contains cycles of length $2k$ for any $m \geq 2$ and any $k$ with $k \neq 5$, $4 \leq k \leq 4p + 1$. 
Cycles in linearized Wenger graphs

- What are the lengths of the cycles in linearized Wenger graphs?
What are the lengths of the cycles in linearized Wenger graphs?

Theorem 1 (Wang (2017))

Let $q$ be the power of prime $p$ with $p \geq 3$. For any integer $k$ with $3 \leq k \leq p^2$, $L_m(q)$ contains cycles of length $2k$. 
The idea of constructing cycles in $L_m(q)$

Embedding cycles in partial planes to get even cycles of length from 6 to $2p^2 - 2p + 2$ in $L_1(p)$.

Constructing cycle of length $2p^2$ in $L_1(p)$ and connecting some points and lines to get cycles of length from $2p^2 - 2p$ to $2p^2$.

Cycles of all even length from 6 to $2p^2$ in $L_1(p)$.

Cycles of all even length from 6 to $2p^2$ in $L_m(q)$.
Embedding cycles in partial planes

Let \( p \) be a prime and \( \mathbb{F}_p \) be the finite field of \( p \) elements. Let \( O \) be the point \((0, 0)\) of a partial plane \( \pi \) which is constructed from projective plane \( PG(2, p) \), and let \( l_0, l_1, ..., l_{p-1} \) be the lines through point \( O \) in \( \pi \).
Embedding cycles in partial planes

- Let $p$ be a prime and $\mathbb{F}_p$ be the finite field of $p$ elements. Let $O$ be the point $(0, 0)$ of a partial plane $\pi$ which is constructed from projective plane $PG(2, p)$, and let $l_0, l_1, \ldots, l_{p-1}$ be the lines through point $O$ in $\pi$.

- Here we take all the points on lines $l_0, l_1, \ldots, l_{p-1}$, and denote the point $p$ by $(x, y)$ with $x, y \in \mathbb{F}_p$ and the line $l_k$ by $[k, 0]$ with $k \in \mathbb{F}_p$. The point $(x, y)$ is on the line $l_k$ if and only if $y = kx$. 
Embedding cycles in $L_1(p)$

Embedding cycles in partial planes

- Let $p$ be a prime and $\mathbb{F}_p$ be the finite field of $p$ elements. Let $O$ be the point $(0, 0)$ of a partial plane $\pi$ which is constructed from projective plane $PG(2, p)$, and let $l_0, l_1, \ldots, l_{p-1}$ be the lines through point $O$ in $\pi$.

- Here we take all the points on lines $l_0, l_1, \ldots, l_{p-1}$, and denote the point $p$ by $(x, y)$ with $x, y \in \mathbb{F}_p$ and the line $l_k$ by $[k, 0]$ with $k \in \mathbb{F}_p$. The point $(x, y)$ is on the line $l_k$ if and only if $y = kx$.

- We use $l_i + p$ to denote the line parallel to $l_i$ that passes through $p$. 
Embedding cycles in $L_1(p)$

- Let $p$ be a prime and $\mathbb{F}_p$ be the finite field of $p$ elements. Let $O$ be the point $(0, 0)$ of a partial plane $\pi$ which is constructed from projective plane $PG(2, p)$, and let $l_0, l_1, \ldots, l_{p-1}$ be the lines through point $O$ in $\pi$.

- Here we take all the points on lines $l_0, l_1, \ldots, l_{p-1}$, and denote the point $p$ by $(x, y)$ with $x, y \in \mathbb{F}_p$ and the line $l_k$ by $[k, 0]$ with $k \in \mathbb{F}_p$. The point $(x, y)$ is on the line $l_k$ if and only if $y = kx$.

- We use $l_i + p$ to denote the line parallel to $l_i$ that passes through $p$.

- For prime $p$, we take a primitive element $\mu$ in $\mathbb{F}_p$ and let $\gamma = \frac{\mu}{\mu - 1} \in \mathbb{F}_p$. Pick any point $p_0$ on $l_0$, different from $O$. Let $p_{i+1}$ be the point of intersection of $l_{i+\gamma} \pmod{p} + p_i$ and $l_{i+1} \pmod{p}$, for all $i = 0, 1, \ldots, p - 2$. 
Example for $p = 5$

Figure 1: Two disjoint paths for $p = 5$
**Example for \( p = 5 \)**

**Figure 1:** Two disjoint paths for \( p = 5 \)

**Lemma 2**

Let \( p_0 \neq p'_0 \in l_0 \) and let \( \Gamma_1, \Gamma_2 \) be two distinct paths with \( p_0 \in \Gamma_1, p'_0 \in \Gamma_2 \), then \( \Gamma_1, \Gamma_2 \) share neither points nor lines.
Lemma 3

Let $p$ be an odd prime. For any integer $k$ with $3 \leq k \leq p^2 - p + 1$, cycles of length $k$ can be embedded in $\pi$. 
Lemma 3

Let $p$ be an odd prime. For any integer $k$ with $3 \leq k \leq p^2 - p + 1$, cycles of length $k$ can be embedded in $\pi$.

Lemma 4

For odd prime $p$, $L_1(p)$ contains cycles of length $2k$ with $3 \leq k \leq p^2 - p + 1$. 
Linearized Wenger graphs

Main results

Future work

Constructing cycles in $L_1(p)$

Notations

- Let $P_2$ and $L_2$ denote the point set and line set in $L_1(p)$, respectively. And let $p_1l_1p_2l_2 \ldots p_{p^2}l_{p^2}$ be a walk of length $2p^2$ in $L_1(p)$ with $p_i \in P_2$ and $l_i \in L_2$, $1 \leq i \leq p^2$. 
Notations

- Let $P_2$ and $L_2$ denote the point set and line set in $L_1(p)$, respectively. And let $p_1 l_1 p_2 l_2 \ldots p_{p^2} l_{p^2}$ be a walk of length $2p^2$ in $L_1(p)$ with $p_i \in P_2$ and $l_i \in L_2$, $1 \leq i \leq p^2$.

- For $p_i \in P_2$ and $l_i \in L_2$ with $1 \leq i \leq p^2$ and $1 \leq j \leq m + 1$, denote $p_i(j)$ the $j$th component of point $p_i$ and $l_i(j)$ the $j$th component of line $l_i$. 
• We take the first components of \( p_i \) and \( l_i \) in table form.
We take the first components of $p_i$ and $l_i$ in table form.

<table>
<thead>
<tr>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>$p$</th>
<th>$p + 1$</th>
<th>$p + 2$</th>
<th>...</th>
<th>$2p$</th>
<th>...</th>
<th>$p^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_i(1)$</td>
<td>1</td>
<td>2</td>
<td>...</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>...</td>
<td>1</td>
<td>...</td>
<td>$p - 1$</td>
</tr>
<tr>
<td>$l_i(1)$</td>
<td>1</td>
<td>2</td>
<td>...</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>...</td>
<td>1</td>
<td>...</td>
<td>$p - 1$</td>
</tr>
</tbody>
</table>

Table 1: The first components of $p_i$ and $l_i$
• We take the first components of $p_i$ and $l_i$ in table form.

<table>
<thead>
<tr>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>$p$</th>
<th>$p+1$</th>
<th>$p+2$</th>
<th>...</th>
<th>$2p$</th>
<th>...</th>
<th>$p^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_i(1)$</td>
<td>1</td>
<td>2</td>
<td>...</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>...</td>
<td>1</td>
<td>...</td>
<td>$p-1$</td>
</tr>
<tr>
<td>$l_i(1)$</td>
<td>1</td>
<td>2</td>
<td>...</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>...</td>
<td>1</td>
<td>...</td>
<td>$p-1$</td>
</tr>
</tbody>
</table>

Table 1: The first components of $p_i$ and $l_i$

• $p_1l_1p_2l_2 \ldots p_{p^2}l_{p^2}$ is a cycle of length $2p^2$. 
• We take the first components of $p_i$ and $l_i$ in table form.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>$p$</th>
<th>$p + 1$</th>
<th>$p + 2$</th>
<th>...</th>
<th>$2p$</th>
<th>...</th>
<th>$p^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_i(1)$</td>
<td>1</td>
<td>2</td>
<td>...</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>...</td>
<td>1</td>
<td>...</td>
<td>$p - 1$</td>
</tr>
<tr>
<td>$l_i(1)$</td>
<td>1</td>
<td>2</td>
<td>...</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>...</td>
<td>1</td>
<td>...</td>
<td>$p - 1$</td>
</tr>
</tbody>
</table>

**Table 1: The first components of $p_i$ and $l_i$**

• $p_1 l_1 p_2 l_2 \ldots p_{p^2} l_{p^2}$ is a cycle of length $2p^2$.

**Lemma 5**

*For odd prime $p$, $L_1(p)$ is Hamiltonian.*
Connecting points and lines in $C_{2p^2}$

- $l_{p-2}$ is adjacent to $p_{(i+1)p-4}$, and $l_{p-1}$ is adjacent to $p_{(i+1)p-3}$, for $i = 1, 2, \ldots, p - 1$.
- $l_{p^2-2}$ is adjacent to $p_{p-4}$, and $l_{p^2-1}$ is adjacent to $p_{p-3}$.

Figure 2: cycles in $L_1(p)$
All even cycles in $L_1(p)$

1. For odd prime $p$, $L_1(p)$ contains cycles of length $2k$, where $3 \leq k \leq p^2 - p + 1$. 
All even cycles in $L_1(p)$

1. For odd prime $p$, $L_1(p)$ contains cycles of length $2k$, where $3 \leq k \leq p^2 - p + 1$.

2. For odd prime $p$, $L_1(p)$ contains cycles of length $2k$, where $p^2 - p \leq k \leq p^2$. 
All even cycles in $L_1(p)$

1. For odd prime $p$, $L_1(p)$ contains cycles of length $2k$, where $3 \leq k \leq p^2 - p + 1$.

2. For odd prime $p$, $L_1(p)$ contains cycles of length $2k$, where $p^2 - p \leq k \leq p^2$.

Lemma 6

For odd prime $p$ and any integer $k$ with $3 \leq k \leq p^2$, $L_1(p)$ contains cycles $C_{2k}$. 
Cycles in $L_m(q)$

Lemma 7

For $m \geq 1$ and odd prime $p$, $L_m(p)$ consists of $p^{m-1}$ components each isomorphic to $L_1(p)$. 
Cycles in $L_m(q)$

**Lemma 7**

For $m \geq 1$ and odd prime $p$, $L_m(p)$ consists of $p^{m-1}$ components each isomorphic to $L_1(p)$.

**Theorem 8**

Let $q$ be the power of prime $p$ with $p \geq 3$. For any integer $k$ with $3 \leq k \leq p^2$, $L_m(q)$ contains cycles of length $2k$. 
Future work

- Find more cycles in Wenger graphs and linearized Wenger graphs.
Future work

- Find more cycles in Wenger graphs and linearized Wenger graphs.

**Conjecture 1**

For every $n \geq 1$, and every prime power $q$, $q \geq 3$, $W_n(q)$ contains cycles of length $2k$, where $4 \leq k \leq q^{n+1}$ and $k \neq 5$. 
Thank you!