List colourings and preference orders

in honour of Haemers-Lazebnik-Woldar

Andrew Thomason (with Arès Méroueh)

8th August 2017

Open question

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Eg $n = 6k$, triples $\{1 + 3j, 3k + 2 + 3j, 6k - 6j\}$ and $\{2 + 3j, 3k + 1 + 3j, 6k - 6j - 3\}$, $0 \leq j < k$, each have sum $3n/2 + 3$ or $3n/2$.

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Can we partition into triples of (roughly) the same *product* p?

If so, $p^{n/3} \approx \prod_{i=1}^{n} i$ so $p \approx (\mathrm{GM}([n]))^3 \approx (n/e)^3$.

Hence NO WE CAN'T,
because triple with 1 in it has product $\leq 1 \cdot (n - 1) \cdot n < n^2$.

Open question

If m is large enough that $mn^2 > (\mathrm{GM}(\{m, m+1, \ldots, n\}))^3$, can we partition $\{m, m+1, \ldots, n\}$ into triples with similar products?

Conjecture

Let $\gamma = 0.116586\ldots$ be the root of $(1/\gamma)^{(1+2\gamma)/3} = e^{1-\gamma}$.

Then the set $\{\lfloor \gamma n \rfloor, \lfloor \gamma n \rfloor + 1, \ldots, n\}$ can be split into triples whose products differ by $o(n^3)$.

Vertex colouring

Let G be a graph or r-uniform hypergraph (edges are r-sets).

A *vertex colouring* of G is a map

$$c : V(G) \to \{\text{colours}\} \quad \text{such that} \quad \text{no edge is monochromatic}$$

An edge is monochromatic if every vertex in it has the same colour.
Here $\{\text{colours}\}$ is the palette of available colours.

The *chromatic number* of G is

$$\chi(G) = \min\{k : \text{ there is a colouring } c : V(G) \to \{1, \dots, k\}\}$$

List colouring

Suppose now we assign a *list* of colours to each vertex, ie

$$L : V(G) \to \mathcal{P}(\{\text{colours}\})$$

We say G is *L-choosable* if there is a vertex colouring

$$c : V(G) \to \{\text{colours}\} \quad \text{with} \quad c(v) \in L(v) \text{ for all v}$$

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$$\chi_l(G) = \min\{k : \; G \text{ is } L\text{-chooseable whenever } \; \forall v \;\; |L(v)| \geq k\}$$

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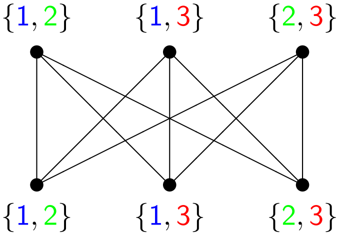
Clearly $\chi_l(G) \geq \chi(G)$ (make $L(v)$ same $\forall v$)

Introduced by Vizing (1976) and by Erdős, Rubin, Taylor (1979)

χ_l can be bigger than χ

$K_{3,3}$ not 2-choosable: $\chi = 2$, $\chi_l \geq 3$



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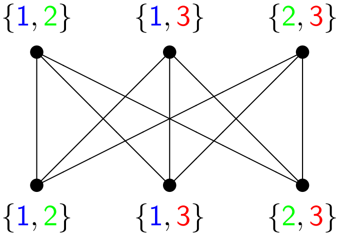
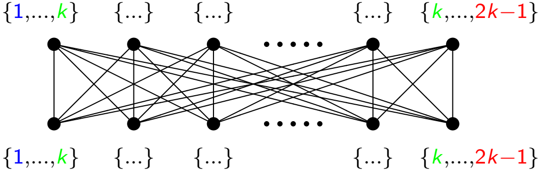


More generally, $K_{m,m}$ is not k-choosable if $m \geq \binom{2k-1}{k}$



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For all graphs G of average degree d, $\chi_l(G) \geq (\frac{1}{2} + o(1)) \log_2 d$

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But how do we show G is not L-chooseable?

Theorem (Sapozhenko '90s)

Let G be a d-regular graph with n vertices. Then there is a collection $\mathcal{C}$ of vertex subsets, called containers, such that

- *for every independent I there is a $C \in \mathcal{C}$ with $I \subset C$*
- $|C| \leq (1/2 + \epsilon)n$ *for all* $C \in \mathcal{C}$
- $|\mathcal{C}| \leq 2^{cn/d}$ *where* $c = c(\epsilon)$

Simple hypergraphs

Simple or *linear* hypergraph: $|e \cap f| \leq 1$ for all distinct edges e, f

A *Steiner triple system* is a simple regular 3-uniform hypergraph

A *Latin square* graph is a simple d-regular subgraph of $K^{(3)}_{d,d,d}$

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Theorem (Saxton+T 12,14 {c.f. Balogh+Morris+Samotij})

Let G be a simple d-regular r-uniform hypergraph with n vertices. Then there is a collection $\mathcal{C}$ of vertex subsets called containers, *such that*

- *for every independent I there is a $C \in \mathcal{C}$ with $I \subset C$*
- *for every $C \in \mathcal{C}$, $|C| \leq (1 - c)|V|$* — *where $c = 1/4r^2$*
- $|\mathcal{C}| \leq 2^{\tau n}$ — *where $\tau = d^{-1/(2r-1)}$*

A lower bound on χ_l

Theorem (Saxton+T 12,14)

Let G be simple (ie linear) r-uniform d-regular. Then

$$\chi_l(G) \geq \left(\frac{1}{r-1} + o(1)\right) \log_r d$$

(bounds too for non-regular, non-simple)

How good is this bound? Are there tight examples?

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So let's try to colour simple regular 3-partite 3-uniform hypergraphs

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Example: $r = 2$ $m = 2k$

$$\begin{array}{cc} 2k & 1 \\ 2k-1 & 2 \\ \vdots & \vdots \\ k+1 & k-1 \\ k & k \\ k-1 & k+1 \\ \vdots & \vdots \\ 2 & 2k-1 \\ 1 & 2k \end{array}$$

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$$\begin{array}{ccc}
3k & k & 2k \\
3k-1 & k-1 & 2k-1 \\
\vdots & \vdots & \vdots \\
2k+1 & 1 & k+1 \\
2k & 3k & k \\
2k-1 & 3k-1 & k-1 \\
\vdots & \vdots & \vdots \\
k+1 & 2k+1 & 1 \\
1 & k+1 & 2k+1 \\
2 & k+2 & 2k+2 \\
\vdots & \vdots & \vdots \\
k & 2k & 3k
\end{array}$$

A colouring algorithm for r-partite hypergraphs

Let G be r-partite r-uniform, average degree d

$$V(G) = V_1 \cup V_2 \cup \cdots \cup V_r, \quad |V_i| = n$$

Let $L : V(G) \to \mathcal{P}(\{\text{colours}\})$, with list sizes ℓ

ALGORITHM:

Randomly number the colours 1 to m (the size of the palette)

Choose an "optimal" preference order on $[m]$

If $v \in V_i$ let $c(v)$ be colour in $L(v)$ most preferred by ith order

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$\Pr(v \in V_i \text{ is green }) = x_i^\ell, \quad x_i =$ "height" of green in ith order

so if $X =$ green vertices then $\mathbb{E}|X_i| = x_i^\ell n$

Preference orders - values

We can assign a *value* to a preference order, namely

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Define $f(r)$ to be the minimum value of all pref orders (as $m \to \infty$)

$f(2) = 1/2$ $\quad$ $f(3) = 1/9$ $\quad$ $f(4) = 0.0262\ldots$ $\quad$ $f(r) = ?$

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Why is this value relevant?

r-partite hypergraphs

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(D_r) if $\prod_{i \neq j} |X_i| \leq n^{r-1}/d$ then $G[X]$ is $\frac{4 \log d}{\log \log d}$-degenerate

(N_r) if $\prod_{i \neq j} |X_i| \geq n^{r-1}/d \times \log^2 d$ then X is not independent.

Almost all G satisfy both (D_r) and (N_r).

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Theorem (Méroueh+T)

If r-partite G satisfies (D_r) and (N_r) then

$$\chi_I(G) \sim g(r) \log_r d$$

where $g(r) = -1/\log_r f(r)$

A *lower* bound for χ_l using preference orders

Let G be r-partite r-uniform, average degree d

Choose random lists $L : V(G) \to \mathcal{P}(\{\text{colours}\})$ each of size ℓ

Suppose G has a colouring $c : V(G) \to \{\text{colours}\}$

Define a preference order on $\{\text{colours}\}$, the r orders being the order of popularity of $c(v)$ in $V_1, V_2, \ldots, V_r$

This preference order has value at least $f(r)$ ...

... some colour (green, say) has positions x_i with $\prod_{i \neq i_x} x_i \geq f(r)$

... information about the size of the green independent set

If G has *(I_r)* this gives a contradiction when $\ell < g(r) \log_r d$

How the numbers stack up

(Saxton+T 12,14) $\forall$ simple d-regular G, $\chi_l(G) \gtrsim \frac{1}{(r-1)} \log_r d$

Méroueh+T $\forall$ r-partite G, (D_r) & (N_r) $\Rightarrow \chi_l(G) \sim g(r) \log_r d$

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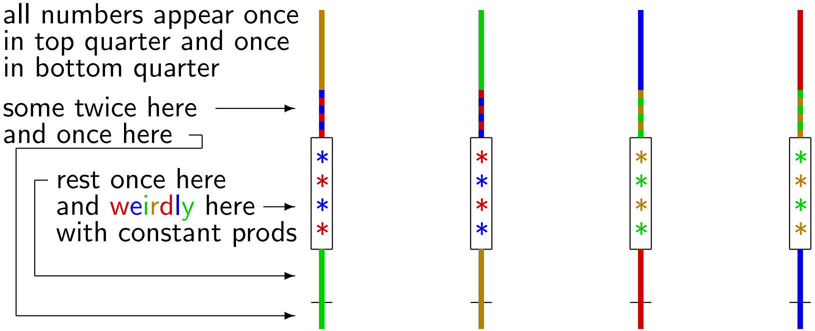
This shows the container argument is tight for $r = 2$ and $r = 3$

but (probably) not for $r \geq 4$

So in fact how big is $f(r)$?

$$\frac{1}{38.13} < f(4) < \frac{1}{38.12}$$



And $\left(\frac{r-1}{re}\right)^{r-1} \leq f(r) \leq \frac{(r-1)!}{r^{r-1}}$ so $g(r) \sim \frac{\log r}{r}$ c.f. container $\frac{1}{r-1}$

Thanks for your attention . . .