On some problems in combinatorics, graph theory and finite geometries

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My plan for today:

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1. Maximum number of λ-colorings of (v, e)-graphs

2. Covering finite vector space by hyperplanes

3. Figures in finite projective planes

4. Hamiltonian cycles and weak pancyclicity

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Problem Let v, e, λ be positive integers.

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of proper vertex colorings in (at most) λ colors a graph with v vertices and e edges can have?

On which graphs is this maximum attained?

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On which graphs is this maximum attained?

The question can be rephrases as the question on maximizing $\chi(G, \lambda)$ over all graphs with v vertices and e edges.

This problem was stated independently by Wilf (82) and Linial (86), and is still largely unsolved.

Maximum number of λ-colorings of (v, e)-graphs.

For every (v, e)-graph G, color its vertices uniformly at random in at most λ colors. What is the maximum probability that a graph is colored properly? On which graph we have the greatest chance to succeed?

$$\mathrm{Prob}(G \text{ is colored properly}) = \frac{\chi(G, \lambda)}{\lambda^v}$$

$$\max\{\mathrm{Prob}(G \text{ is colored properly})\} = \frac{f(v, e, \lambda)}{\lambda^v}$$

Maximum number of λ-colorings of (v, e)-graphs.

Problem. Is it true that there exists p_0 such that

$$f(2p, p^2, \lambda) = \chi(K_{p,p}, \lambda)$$

for all $p \geq p_0$ and all $\lambda \geq 2$, and $K_{p,p}$ is the only extremal graph?

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Known to be true for $\lambda = 2, 3, 4$, and $\lambda \geq p^5$.

What if $5 \leq \lambda < p^5$ **???**

Maximum number of λ-colorings of (v, e)-graphs.

$f(2p, p^2, 2) = \chi(K_{p,p}, 2)$. – trivial.

$f(2p, p^2, \lambda) = \chi(K_{p,p}, \lambda)$ if $\lambda \geq p^5$. – FL (91)

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$f(2p, p^2, 3) = \chi(K_{p,p}, 3)$.

$f(2p, p^2, 4) \sim \chi(K_{p,p}, 4) \sim (6 + o(1))4^p$, as $p \to \infty$. –
FL - O. Pikhurko - A. Woldar (07)

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Maximum number of λ-colorings of (v, e)-graphs.

Known:

- $f(v, e, 2)$: FL (89)
- $f(v, e, 3)$: bounds

  FL (89, 90, 91), R. Liu (93), K. Dohmen (93, 98), X.B. Chen (96), O. Byer (98), I. Simonelli (08), S. Norine (11)
- For $0 \leq e \leq v^2/4$, it was conjectured (FL (91)) that

$$f(v, e, 3) = \chi(K_{a,b,p}, 3),$$

  where $K_{a,b,p}$ is semi-complete bipartite graph: $v = a + b + 1$, $e = ab + p$, $0 \leq p \leq a \leq b$.

  It was proven for sufficiently large e by P.-S. Loh - O. Pikhurko - B. Sudakov (10)

Maximum number of λ-colorings of (v, e)-graphs.

Let $e = e(T_{r,v}) =: t_{r,v}$.

- If $\lambda \geq 2\binom{t_{r,v}}{3} + 1$, then $f(v, t_{r,v}, \lambda) = \chi(T_{r,v}, \lambda)$, and $T_{r,v}$ is the only extremal graph. – FL (91)

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- Fix $r \geq 3$. For all sufficiently large v ,

$$f(v, t_{r,v}, r+1) = \chi(T_{r,v}, r+1),$$

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- Fix $r \geq 2$. For all v $(v \geq r)$,

$$f(v, t_{r,v}, r+1) = \chi(T_{r,v}, r+1),$$

  and $T_{r,v}$ is the only extremal graph. – FL - S.Tofts (10)

Maximum number of λ-colorings of (v, e)-graphs.

- Fix λ and r so that $\lambda > r \geq 2$ and r divides λ. For all sufficiently large v,

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  Conjecture: For all λ and r, $\lambda \geq r \geq 2$,

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- For fixed sufficiently large r and λ, $\lambda \geq 100 \frac{r^2}{\log r}$, $T_{r,v}$ is asymptotically extremal:

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The Conjecture is FALSE! – J. Ma - H. Naves (15)

- (i) For any integers $r \geq 50000$, there exists λ such that

$$19r \leq \lambda \leq \frac{r^2}{200 \log r} - r,$$

and the Conjecture is false.

(ii) If $13 \leq r + 3 \leq \lambda \leq 2r - 7$, the Conjecture is false.

2. Covering finite vector space by hyperplanes

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$\mathbb{F}_q$ is a finite field of $q = p^e$ elements, p is prime.

$e_1, e_2, \dots e_n$ – the standard basis of $\mathbb{F}_q^n$.

$a_1, a_2, \dots a_n$ – any basis of $\mathbb{F}_q^n$.

For $0 \neq x \in \mathbb{F}_q^n$, $x^\perp$ is the orthogonal complement of x in $\mathbb{F}_q^n$ with respect to the standard inner product in $\mathbb{F}_q^n$.

Rename the sequence

$$e_1, e_2, \dots, e_n, a_1, a_2, \dots, a_n$$

as

$$b_1, b_2, \dots, b_n, b_{n+1}, b_{n+2}, \dots, b_{2n}.$$

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N. Alon - M. Tarsi (89)

Covering finite vector space by hyperplanes

Let A be an $n \times n$ matrix over $\mathbb{F}_q$, q is a prime power.

A vector $x \in \mathbb{F}_q^n$ is called **good for** A, or **nowhere-zero for** A, if both x and Ax have no zero components. If x is good for A, we also say that A **has a good vector** x.

Question (F. Jaeger (81)):

Consider an n-dimensional vector space over $\mathbb{F}_5$. Is it true that for any two bases B_1 and B_2, there exists a hyperplane H which is disjoint from $B_1 \cup B_2$?

The question is equivalent to the following:

Does every $A \in GL(n, \mathbb{F}_5)$, $n \geq 3$, have a good vector?

Covering finite vector space by hyperplanes

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YES, if the prime power q is NOT a prime!

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What if $q = p$ is prime?

A - T Conjecture N. Alon - M. Tarsi (89):

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- A simple observation:

A - T Conjecture holds for primes p such that $4 \leq n + 1 \leq p$.
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What happens if when $n > p$?

Covering finite vector space by hyperplanes

• If A is chosen uniformly from $GL(n, \mathbb{F}_p)$, A - T Conjecture holds almost surely as $n \to \infty$.
(N. Alon - unpublished)

• A - T Conjecture holds if $n \leq 2^{p-2} - 1$. – Y. Yu (99)

So it is true for

$p = 5$ and $n \leq 7$,
$p = 7$ and $n \leq 31$,
$p = 11$ and $n \leq 511$.

Covering finite vector space by hyperplanes

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Let $P(A, q)$ denote the number of good vectors of A.

Recall that for each $A \in GL_n(\mathbb{F}_q)$, we have two bases of $\mathbb{F}_q^n$:

$$\{b_i = e_i, i = 1 \ldots, n\} \quad \text{and} \quad \{b_{n+i} = a_i, i = 1, \ldots, n\},$$

where a_i is the i-th row of A.
Therefore

$$P(A, q) = \left|\overline{\bigcup_{j=1}^{2n} b_j^{\perp}}\right|.$$

Covering finite vector space by hyperplanes

Fix q and n. What is the minimum value of $P(A, q)$ over all $A \in GL(n, \mathbb{F}_q)$?

Covering finite vector space by hyperplanes

Fix q and n. What is the minimum value of $P(A, q)$ over all $A \in GL(n, \mathbb{F}_q)$?

Let $n = 2k$. Consider the following matrix:

$$A^{\star} = \begin{pmatrix} A_1 & 0 & 0 & \ddots & 0 \\ 0 & A_2 & 0 & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \ddots & 0 & A_k \end{pmatrix},$$

where $A_i \in GL(2, q)$ with no zero entries. Note that

$$P(A^{\star}, q) = [(q-1)(q-3)]^k$$

Covering finite vector space by hyperplanes

R. Baker - J. Bonin – FL– E. Shustin (94)

For $n = 2k \geq 4$ and $q \geq 2\binom{2n}{3} + 1$

$$P(A, q) \geq P(A^\star, q),$$

with the equality if and only if A can be transformed to $A^\star$ by some permutations of its rows and columns.

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Recall a result on the maximum number of colorings :

If $\lambda \geq 2\binom{t_{r,v}}{3} + 1$, then

$$f(v, t_{r,v}, \lambda) = \chi(T_{r,v}, \lambda),$$

and $T_{r,v}$ is the only extremal graph.

Covering finite vector space by hyperplanes

To show the extremality of the construction, in both cases of

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the Whitney's Broken Circuits Theorem was used.

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Positive answer, of course, implies A - T Conjecture in a strong way.

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What is a projective plane of order $r \geq 2$? We will denote it π_r.

- Axiomatic definition as a incidentce system on points and lines.
- A $2 - (r^2 + r + 1, r + 1, 1)$ SBIBD.
- A bipartite $(r + 1)$-regular graph of diameter 3 and girth 6.

  Or, an incidence system of points and lines whose Levi graph is $(r + 1)$-regular graph of diameter 3 and girth 6.

Figures in finite projective planes

A MODEL of a projective plane π_r:

- $r = q$ – prime power,
- Points: 1-dim subspaces (points) in F_q^3,
- Lines: 2-dim subspaces in F_q^3,
- Incidence: containment

This projective plane has order q, it is denoted by $PG(2,q)$, and is called the **classical** plane of order q.

- π_r are known to exist for all $r = q$ – prime power.
- No example with r being not a prime power is known.
- For $q \geq 9$, there are non-classical π_q.
- For $r = p$ – prime, no example of a non-classical π_p is known.

Figures in finite projective planes

A **partial plane** is an incidence system of points and lines such that any two distinct points are on at most one line.

The definition implies that in a partial plane any two distinct lines share at most one point.

A projective plane is a partial plane.

Levi graph of a partial plane is a bipartite graph without 4-cycles.

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Levi graph of a partial plane is a bipartite graph without 4-cycles.

We say that a partial plane π^1 can be **embedded** into a partial plane π^2 if there exists an injective map of the set of points of π^1 to the set of points of π^2 such that colinear points are mapped to colinear points.

Equivalently: $Levi(\pi^1)$ is isomorphic to a subgraph of $Levi(\pi^2)$.

Figures in finite projective planes

Problem: Given a finite partial plane π. Is there a finite projective plane π_r such that π can be embedded in π_r?

Equivalently, given a finite bipartite graph without 4-cycles, is it isomorphic to a subgraph of the Levi graph of a finite projective plane?

P. Erdős (79), D. Welsh (76), M. Hall (???)

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• Can Pappus configurartion be found in every finite projective plane of order $r \geq 3$???

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- Does every π_r contain a k-gon for every k, $3 \leq k \leq r^2 + r + 1$?

Figures in finite projective planes

• Does every π_r contain a k-gon for every k, $3 \leq k \leq r^2 + r + 1$?
YES! – FL - K. Mellinger - O. Vega (13)

A trivial thought: Suppose we have a 4-cycle-free (m, n)-bipartite graph G which contains more copies of a certain subgraph H than any (m, n)-subgraph of the Levi graph of any π_r. Then H cannot be embedded in $Levi(\pi_r)$.

- $Levi(\pi_r)$ has more edges than any other 4-cycle-free graph G with the same partition sizes. – I. Reiman (58)
- $Levi(\pi_r)$ has more 6-cycles than any other 4-cycle-free graph G with the same partition sizes. – FL - G. Fiorini (98)
- $Levi(\pi_r)$ has more 8-cycles than any other 4-cycle-free graph G with the same partition sizes if $r \geq 13$. – FL - S. De Winter - J. Verstraëte (08) What about $2k$-cycles for $k \geq 5$???

Figures in finite projective planes

The problems above led to the question of counting the number of $2k$-cycles in $Levi(\pi_r)$.
Let $c_{2k}(\pi_r)$ denote the number of $2k$-cycles in $Levi(\pi_r)$.

Explicit formuli for $c_{2k}(\pi_r)$ exists for

$k = 3, 4, 5, 6$ – FL - K. Mellinger - O. Vega (09)

$k = 7, 8, 9, 10$ – A. Voropaev (13)

In all these cases $c_{2k}(\pi_r)$ depend on k and on r ONLY!

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QUESTION: Is it true that there exists $k \geq 11$, such that

$$c_{2k}(\pi_q) \neq c_{2k}(PG(2, q)) \ ?$$

4. Hamiltonian cycles and weak pancyclicity

Let $f, g : \mathbb{N} \to (0, \infty)$. We write

$$f = o_n(g) = o(g), \text{ if } f/g \to 0, \ n \to \infty.$$

Let (n_i) be a sequence of positive integers, $n_i \to \infty$.

Let $\Gamma_i = (V_{n_i}, E_{n_i})$ – a sequence of simple graphs, $|V_{n_i}| = n_i$.
If

$$|E_{n_i}| = o_i({n_i}^2)$$

and say that Γ_i forms a sequence of sparse graphs.

If Γ_i is d_i-regular, (Γ_i) sparse iff $d_i = o_i(n_i)$.

Example. $Levi(\pi_q)$ – Levi graph of a projective plane π_q of order q (bipartite point-line incidence graph of π_q):

Then $n_q = 2(q^2 + q + 1)$, q is a prime power,
$d_q = q + 1 = o_q(n_q)$: $Levi(\pi_q)$ is sparse.

Hamiltonian cycles and weak pancyclicity

Γ is hamiltonian if it contains a spanning cycle (= hamiltonian cycle).

G. Dirac (1952): Let Γ have n vertices, $n \geq 3$. If $d_\Gamma(x) = d(x) \geq n/2$ for every vertex x of Γ, then Γ is hamiltonian.

O. Ore (1960): Let Γ have n vertices, $n \geq 3$. If $d(x) + d(y) \geq n$ for every pair of non-adjacent vertices x and y, then Γ is hamiltonian.

Closure $cl(\Gamma)$ is a graph obtained from Γ by repeatedly adding a new edge xy, connecting a nonadjacent pair of vertices x and y such that $d(x) + d(y) \geq n$.

A. Bondy - V. Chvátal (1972): Γ is hamiltonian iff $cl(\Gamma)$ is hamiltonian.

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None of these theorems implies that $Levi(\pi_q)$ is hamiltonian.

Hamiltonian cycles and weak pancyclicity

Is $Levi(\pi_q)$ is hamiltonian?

J. Singer (1938): Yes, if $\pi_q = PG(2, q)$ – the classical plane.

E. Schmeichel (1989): Yes, for $\pi_p = PG(2, p)$ (different cycle).

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Moreover, $Levi(\pi_r)$ contains a cycle of length $2k$ for every k, $3 \leq k \leq r^2 + r + 1$. (weakly pancyclic)

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$Levi(\pi_r)$ is also known as:

• a $(r+1, 6)$-cage;

• a bipartite 4-cycle-free graph with partitions of size $r^2 + r + 1$ and having the maximum number of edges;

• a generalized 3-gon of order r: a bipartite $(r+1)$-regular graph of diameter 3 and girth 6.

Hamiltonian cycles and weak pancyclicity

A generalized d-gon of order r is a geometry with Levi graph being a bipartite $(r+1)$-regular graph of diameter d and girth $2d$.

Hamiltonian cycles and weak pancyclicity

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Existence for $r \geq 2$ and $d \geq 3$:

J. Tits (1959) for $d \in \{4, 6\}$ $r = q$ – any prime power.

W. Feit and G. Higman (1964): If $r \geq 2$ and $d \geq 3$, it can exists only for $d \in \{3, 4, 6\}$.

Here are equivalent definitions:

- a $(r+1, 2d)$-cage;
- a bipartite graph of girth at least $2d$ with partitions of size $r^{d-1} + r^{d-2} + \cdots + r + 1$ and having the maximum number of edges.

Hamiltonian cycles and weak pancyclicity

Denote a generalized $(r+1)$-regular d-gon by π_r^d, $d = 3, 4, 6$.

$\pi_r^3 = \pi_r$ – projective plane of order q.

Easy to show that $|V(Levi(\pi_r^d))| = 2(r^{d-1} + r^{d-2} + \cdots + r + 1)$.
Hence, $Levi(\pi_r^d)$ is sparse.

Is $Levi(\pi_r^d)$ hamiltonian for $d = 4, 6$???

J. Alexander - FL - A. Thomason (2016+): Yes, provided r being sufficiently large.

Is $Levi(\pi_r^d)$ weakly pancyclic for $d = 4, 6$???

Not known. J. Exoo confirmed the weak pancyclicity for $Levi(\pi_3^4)$ and $Levi(\pi_5^4)$.

Hamiltonian cycles and weak pancyclicity

More general questions:

Consider a graph Γ on n vertices with girth at least $2k+1$ and having "MANY" edges.

Is Γ hamiltonian? Is G weakly pancyclic?

Hamiltonian cycles and weak pancyclicity

Let $\mathcal{G}(n,p)$ be a random graph model, and $\Gamma \in \mathcal{G}(n,p)$.

What is Prob [Γ is hamiltonian] ?

L. Posa (1972): If $\Gamma \in \mathcal{G}(n,p)$ and $p = c \log n/n$, then Γ is hamiltonian, i.e.,

$$\text{Prob}\,[\Gamma \text{ is hamiltonian}] \to 1, \;\; n \to \infty,$$

provided that constant c is sufficiently large.

Note that the expected degree of a vertex of Γ is
$(n-1)p \sim c \log n = o(n)$,
and the expected number of edges is $\sim (c/2) n \log n = o(n^2)$.
Hence, Γ is sparse.

Pseudo-random graphs

Pseudo-random graphs

A. Thomason (1987): (p, α)-jumbled graphs.

F. Chung, R. Graham, R. Wilson (1989): quasi-random graphs

M. Krivelevich and B. Sudakov (2006): a survey.

Let $A(\Gamma)$ be the adjacency matrix of Γ. $A(\Gamma)$ is a real symmetric matrix, and so $A(\Gamma)$ has real eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. If Γ is d-regular, then $\lambda_1 = d$. Let

$$\lambda = \lambda(\Gamma) := \max\{|\lambda_i| : i = 2, 3, \ldots, n\}.$$

Let Γ be a d-regular graph on n vertices.

The difference $d - \lambda$ is called the spectral gap.

Hamiltonian cycles and weak pancyclicity

It turns out that the spectral gap $d - \lambda$ is responsible for the pseudo-random properties of graphs:

The larger the spectral gap is, the closer the edge distribution of Γ approaches that of a random graph $\mathcal{G}(n, d/n)$.

It is known that if $d \leq (1 - \epsilon)n$ for some $\epsilon > 0$, then $\lambda \geq c\sqrt{d}$.

If

$$c_1\sqrt{d} < \lambda < c_2\sqrt{d}$$

for some $c_1, c_2 > 0$, then then Γ is a "good" pseudo-random graph.

For $\Gamma \in \mathcal{G}(n, 1/2)$,

$$\lambda(\Gamma) \approx 2\sqrt{n/2} \approx 2\sqrt{d}.$$

Hamiltonian cycles and weak pancyclicity

Theorem (M. Krivelevich, B. Sudakov (2002))

Let Γ be a d-regular n-vertex graph. If n is large enough and

$$\lambda \leq \frac{(\log \log n)^2}{1000 \log n \, (\log \log \log n)} \, d,$$

then Γ is Hamiltonian.

Hamiltonian cycles and weak pancyclicity

Theorem (M. Krivelevich, B. Sudakov (2002))

Let Γ *be a* d*-regular* n*-vertex graph. If* n *is large enough and*

$$\lambda \leq \frac{(\log \log n)^2}{1000 \log n \, (\log \log \log n)} \, d,$$

then Γ *is Hamiltonian.*

If Γ is bipartite, then $\lambda = |-d| = d$, and the condition fails.

E.g., it fails for $Levi(\pi_r^d)$.

Hamiltonian cycles and weak pancyclicity

Theorem (J. Alexander - FL - A. Thomason (2016+))
Let Γ be a d-regular n-vertex <u>*bipartite*</u> *graph, and*

$$\lambda = \lambda(\Gamma) := \max\{|\lambda_i| : i = 2, 3, \ldots, n-1\}.$$

If n is large enough and

$$\lambda \leq \frac{(\log \log n)^2}{2000 \log n \, (\log \log \log n)} \, d,$$

then Γ is Hamiltonian.

This gave another motivation for determining bounds on λ.

Hamiltonian cycles and weak pancyclicity

The following bipartite graphs and some of their subgraphs are hamiltonian when their order is sufficiently large:

- Generalized polygons π_r^d and their biaffine parts
- S. Cioabă - FL - W. Li (2014): Wenger graphs $W_n(q)$
- X. Cao - M. Lu - D. Wan - L.P. Wang - Q. Wang (2015): linearized Wenger graphs $L_n(q)$
- E. Moorehouse - S. Sun - J. Williford (2017): graphs $D(4, q)$

Hamiltonian cycles and weak pancyclicity

Problem 1: Establish similar results for all q, not just sufficiently large.

Problem 2: Strengthen a result of A. Frieze - M. Krivelevich (02) on

the hamiltonicity of random subgraphs of d-regular pseudo-random graphs.

This allows to show existence of smaller cycles in the graph.

At this time it is useful only for $d >> n^{3/4}(\log n)^3$ and large n.

Thank you!