On some problems in combinatorics, graph theory and finite geometries

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My plan for today:
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1. Maximum number of $\lambda$-colorings of $(v, e)$-graphs

2. Covering finite vector space by hyperplanes

3. Figures in finite projective planes

4. Hamiltonian cycles and weak pancyclicity
1. Maximum number of $\lambda$-colorings of $(v, e)$-graphs.

Problem
Let $v, e, \lambda$ be positive integers. What is the maximum number $f(v, e, \lambda)$ of proper vertex colorings in (at most) $\lambda$ colors a graph with $v$ vertices and $e$ edges can have? On which graphs is this maximum attained? The question can be rephrased as the question on maximizing $\chi(G, \lambda)$ over all graphs with $v$ vertices and $e$ edges. This problem was stated independently by Wilf (82) and Linial (86), and is still largely unsolved.
1. Maximum number of \( \lambda \)-colorings of \((v, e)\)-graphs.

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f(v, e, \lambda)
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of proper vertex colorings in (at most) \( \lambda \) colors a graph with \( v \) vertices and \( e \) edges can have?

On which graphs is this maximum attained?
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Maximum number of $\lambda$-colorings of $(v, e)$-graphs.

For every $(v, e)$-graph $G$, color its vertices uniformly at random in at most $\lambda$ colors. What is the maximum probability that a graph is colored properly? On which graph we have the greatest chance to succeed?

$$\text{Prob}(G \text{ is colored properly}) = \frac{\chi(G, \lambda)}{\lambda^v}$$

$$\max\{\text{Prob}(G \text{ is colored properly})\} = \frac{f(v, e, \lambda)}{\lambda^v}$$
Maximum number of $\lambda$-colorings of $(v, e)$-graphs.

Problem. Is it true that there exists $p_0$ such that

$$f(2p, p^2, \lambda) = \chi(K_{p,p}, \lambda)$$

for all $p \geq p_0$ and all $\lambda \geq 2$, and $K_{p,p}$ is the only extremal graph?
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Known to be true for $\lambda = 2, 3, 4$, and $\lambda \geq p^5$.

What if $5 \leq \lambda < p^5$ ???
Maximum number of $\lambda$-colorings of $(v, e)$-graphs.

\[ f(2p, p^2, 2) = \chi(K_{p,p}, 2). \text{ – trivial.} \]

\[ f(2p, p^2, \lambda) = \chi(K_{p,p}, \lambda) \text{ if } \lambda \geq p^5. \text{ – FL (91)} \]
Maximum number of \( \lambda \)-colorings of \((v, e)\)-graphs.

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f(2p, p^2, 3) = \chi(K_{p,p}, 3).
\]

\[
f(2p, p^2, 4) \sim \chi(K_{p,p}, 4) \sim (6 + o(1))4^p, \text{ as } p \to \infty. \quad \text{FL - O. Pikhurko - A. Woldar (07)}
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f(2p, p^2, 4) = \chi(K_{p,p}, 4) \text{ for all sufficiently large } p. - \text{S. Norine (11)}.
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f(2p, p^2, 4) = \chi(K_{p, p}, 4) \text{ for all } p \geq 2. \quad \text{S. Tofts (13)}.
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Maximum number of $\lambda$-colorings of $(v, e)$-graphs.

**Known:**

- $f(v, e, 2)$: FL (89)

- $f(v, e, 3)$: bounds
  - FL (89, 90, 91), R. Liu (93), K. Dohmen (93, 98), X.B. Chen (96), O. Byer (98), I. Simonelli (08), S. Norine (11)

- For $0 \leq e \leq v^2/4$, it was conjectured (FL (91)) that
  \[ f(v, e, 3) = \chi(K_{a,b,p}, 3), \]
  where $K_{a,b,p}$ is semi-complete bipartite graph: $v = a + b + 1$, $e = ab + p$, $0 \leq p \leq a \leq b$.

  It was proven for sufficiently large $e$ by P.-S. Loh - O. Pikhurko - B. Sudakov (10)
Maximum number of $\lambda$-colorings of $(\nu, e)$-graphs.

Let $e = e(T_{r,\nu}) =: t_{r,\nu}$.

- If $\lambda \geq 2^{t_{r,\nu}} + 1$, then $f(\nu, t_{r,\nu}, \lambda) = \chi(T_{r,\nu}, \lambda)$, and $T_{r,\nu}$ is the only extremal graph. – FL (91)
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- Fix $r \geq 3$. For all sufficiently large $v$, $f(v, t_{r,v}, r + 1) = \chi(T_{r,v}, r + 1)$, and $T_{r,v}$ is the only extremal graph.

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- Fix $r \geq 2$. For all $v$ ($v \geq r$),

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  and $T_{r,v}$ is the only extremal graph. – FL - S.Tofts (10)
Maximum number of $\lambda$-colorings of $(v, e)$-graphs.

- Fix $\lambda$ and $r$ so that $\lambda > r \geq 2$ and $r$ divides $\lambda$. For all sufficiently large $v$,

$$f(v, t_{r,v}, \lambda) = \chi(T_{r,v}, \lambda),$$

and $T_{r,v}$ is the only extremal graph. – S. Norine (11)

**Conjecture:** For all $\lambda$ and $r$, $\lambda \geq r \geq 2$,

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- For fixed sufficiently large $r$ and $\lambda$, $\lambda \geq 100 \frac{r^2}{\log r}$, $T_{r,v}$ is asymptotically extremal:

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J. Ma - H. Naves (15)
Maximum number of $\lambda$-colorings of $(v, e)$-graphs.

- For fixed sufficiently large $r$ and $\lambda$, $\lambda \geq 100 \frac{r^2}{\log r}$, $T_{r,v}$ is asymptotically extremal:

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The Conjecture is FALSE! – J. Ma - H. Naves (15)

- (i) For any integers $r \geq 50000$, there exists $\lambda$ such that

$$19r \leq \lambda \leq \frac{r^2}{200 \log r} - r,$$

and the Conjecture is false.

(ii) If $13 \leq r + 3 \leq \lambda \leq 2r - 7$, the Conjecture is false.
2. Covering finite vector space by hyperplanes

$F_q$ is a finite field of $q = p^e$ elements, $p$ is prime.

$e_1, e_2, \ldots, e_n$ – the standard basis of $F_q^n$.

$a_1, a_2, \ldots, a_n$ – any basis of $F_q^n$.

For $0 \neq x \in F_q^n$, $x^\perp$ is the orthogonal complement of $x$ in $F_q^n$ with respect to the standard inner product in $F_q^n$.

Rename the sequence $e_1, e_2, \ldots, e_n, a_1, a_2, \ldots, a_n$ as $b_1, b_2, \ldots, b_n, b_n+1, b_n+2, \ldots, b_{2n}$.

Problem: Let $n \geq 3$ and $q \geq 4$. Is it true that $2^n \bigcup_{i=1}^n b_i^\perp = F_q^n$?

N. Alon - M. Tarsi (89)
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Covering finite vector space by hyperplanes

Let $A$ be an $n \times n$ matrix over $\mathbb{F}_q$, $q$ is a prime power.

A vector $x \in \mathbb{F}_q^n$ is called good for $A$, or nowhere-zero for $A$, if both $x$ and $Ax$ have no zero components. If $x$ is good for $A$, we also say that $A$ has a good vector $x$.

Question (F. Jaeger (81)):

Consider an $n$-dimensional vector space over $\mathbb{F}_5$. Is it true that for any two bases $B_1$ and $B_2$, there exists a hyperplane $H$ which is disjoint from $B_1 \cup B_2$?

The question is equivalent to the following:

Does every $A \in GL(n, \mathbb{F}_5)$, $n \geq 3$, have a good vector?
Let $n \geq 3$ and $q \geq 4$. Is it true that every $A \in GL(n, q)$ has a good vector?
Covering finite vector space by hyperplanes

Let $n \geq 3$ and $q \geq 4$. Is it true that every $A \in \text{GL}(n, q)$ has a good vector?

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Covering finite vector space by hyperplanes

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Covering finite vector space by hyperplanes

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Let \( n \geq 3 \) and \( q \geq 4 \). Is it true that for any two bases \( B_1 \) and \( B_2 \) of \( \mathbb{F}_q^n \), there exists a hyperplane \( H \) which is disjoint from \( B_1 \cup B_2 \)?

YES, if the prime power \( q \) is NOT a prime!

N. Alon - M. Tarsi (89).
What if $q = p$ is prime?

**A - T Conjecture** N. Alon - M. Tarsi (89):

Let $n \geq 3$ and $q = p \geq 4$. Then every $A \in GL(n, p)$ has a good vector.
Covering finite vector space by hyperplanes

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- A simple observation:

  A - T Conjecture holds for primes $p$ such that $4 \leq n + 1 \leq p$.

  R. Baker- J. Bonin – FL– E. Shustin (94)
Covering finite vector space by hyperplanes

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Covering finite vector space by hyperplanes

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Covering finite vector space by hyperplanes

• If $A$ is chosen uniformly from $GL(n, \mathbb{F}_p)$, A - T Conjecture holds almost surely as $n \to \infty$. (N. Alon - unpublished)

• A - T Conjecture holds if $n \leq 2^{p-2} - 1$. – Y. Yu (99)

So it is true for

$p = 5$ and $n \leq 7$,
$p = 7$ and $n \leq 31$,
$p = 11$ and $n \leq 511$. 
Y. Yu proved a more general statement: there exists $x$ with no zero components such that $Ax$ has at most $n/2^{p-2}$ components.
Covering finite vector space by hyperplanes

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What if $n \geq 2^{p-2}$? E.g., $p = 5$, $n \geq 8$?

Maybe another approach can be tried...
Y. Yu proved a more general statement: there exists \( x \) with no zero components such that \( Ax \) has at most \( n/2^{p-2} \) components.

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Maybe another approach can be tried...

Let \( P(A, q) \) denote the number of good vectors of \( A \).

Recall that for each \( A \in \text{GL}_n(\mathbb{F}_q) \), we have two bases of \( \mathbb{F}_q^n \):

\[
\{ b_i = e_i, i = 1 \ldots, n \} \quad \text{and} \quad \{ b_{n+i} = a_i, i = 1, \ldots, n \},
\]

where \( a_i \) is the \( i \)-th row of \( A \).

Therefore

\[
P(A, q) = \left| \bigcup_{j=1}^{2n} b_j^\perp \right|.
\]
Fix $q$ and $n$. What is the minimum value of $P(A, q)$ over all $A \in GL(n, \mathbb{F}_q)$?
Covering finite vector space by hyperplanes

Fix $q$ and $n$. What is the minimum value of $P(A, q)$ over all $A \in GL(n, \mathbb{F}_q)$?

Let $n = 2k$. Consider the following matrix:

$$A^* = \begin{pmatrix}
A_1 & 0 & 0 & \cdots & 0 \\
0 & A_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & A_k
\end{pmatrix},$$

where $A_i \in GL(2, q)$ with no zero entries. Note that

$$P(A^*, q) = [(q - 1)(q - 3)]^k$$
Covering finite vector space by hyperplanes

R. Baker - J. Bonin – FL– E. Shustin (94)

For \( n = 2k \geq 4 \) and \( q \geq 2^{\left(\frac{2n}{3}\right)} + 1 \)

\[
P(A, q) \geq P(A^*, q),
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with the equality if and only if \( A \) can be transformed to \( A^* \) by some permutations of its rows and columns.
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Recall a result on the maximum number of colorings:

If \( \lambda \geq 2^{\left(\frac{t_r}{3}\right)} + 1 \), then

\[ f(v, t_r, \lambda) = \chi(T_{r,v}, \lambda), \]

and \( T_{r,v} \) is the only extremal graph.
Covering finite vector space by hyperplanes

To show the extremality of the construction, in both cases of $\chi(G, \lambda)$ and $P(A, q)$,
the Whitney’s Broken Circuits Theorem was used.
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QUESTION: Is it true that for \( n = 2k \geq 4 \) and every \( q \geq 4 \),

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Positive answer, of course, implies A - T Conjecture in a strong way.
3. Figures in finite projective planes

What is a projective plane of order $r \geq 2$? We will denote it $\pi_r$.

▶ Axiomatic definition as an incidence system on points and lines.

▶ A $2$-$(r^2 + r + 1, r + 1, 1)$ SBIBD.

▶ A bipartite $(r + 1)$-regular graph of diameter 3 and girth 6. Or, an incidence system of points and lines whose Levi graph is $(r + 1)$-regular graph of diameter 3 and girth 6.
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Figures in finite projective planes

A MODEL of a projective plane $\pi_r$:

- $r = q$ – prime power,
- Points: 1-dim subspaces (points) in $F_q^3$,
- Lines: 2-dim subspaces in $F_q^3$,
- Incidence: containment

This projective plane has order $q$, it is denoted by $PG(2, q)$, and is called the **classical** plane of order $q$.

- $\pi_r$ are known to exist for all $r = q$ – prime power.
- No example with $r$ being not a prime power is known.
- For $q \geq 9$, there are non-classical $\pi_q$.
- For $r = p$ – prime, no example of a non-classical $\pi_p$ is known.
Figures in finite projective planes

A partial plane is an incidence system of points and lines such that any two distinct points are on at most one line.

The definition implies that in a partial plane any two distinct lines share at most one point.

A projective plane is a partial plane.

Levi graph of a partial plane is a bipartite graph without 4-cycles.
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**Levi graph of a partial plane is a bipartite graph without 4-cycles.**

We say that a partial plane $\pi^1$ can be **embedded** into a partial plane $\pi^2$ if there exists an injective map of the set of points of $\pi^1$ to the set of points of $\pi^2$ such that colinear points are mapped to colinear points.

Equivalently: $Levi(\pi^1)$ is isomorphic to a subgraph of $Levi(\pi^2)$. 
Figures in finite projective planes

**Problem:** Given a finite partial plane $\pi$. Is there a finite projective plane $\pi_r$ such that $\pi$ can be embedded in $\pi_r$?

Equivalently, given a finite bipartite graph without 4-cycles, is it isomorphic to a subgraph of the Levi graph of a finite projective plane?

P. Erdős (79), D. Welsh (76), M. Hall (???)
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- Can Pappus configuration be found in every finite projective plane of order \( r \geq 3 \) ???
Figures in finite projective planes

- Does every $\pi_r$ contain a $k$-gon for every $k$, $3 \leq k \leq r^2 + r + 1$?
Figures in finite projective planes

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  YES! – FL - K. Mellinger - O. Vega (13)

**A trivial thought:** Suppose we have a 4-cycle-free $(m, n)$-bipartite graph $G$ which contains more copies of a certain subgraph $H$ than any $(m, n)$-subgraph of the Levi graph of any $\pi_r$. Then $H$ cannot be embedded in $Levi(\pi_r)$.

- $Levi(\pi_r)$ has more edges than any other 4-cycle-free graph $G$ with the same partition sizes. – I. Reiman (58)

- $Levi(\pi_r)$ has more 6-cycles than any other 4-cycle-free graph $G$ with the same partition sizes. – FL - G. Fiorini (98)

- $Levi(\pi_r)$ has more 8-cycles than any other 4-cycle-free graph $G$ with the same partition sizes if $r \geq 13$. – FL - S. De Winter - J. Verstraëte (08) What about $2k$-cycles for $k \geq 5$ ???
The problems above led to the question of counting the number of $2k$-cycles in $Levi(\pi_r)$.
Let $c_{2k}(\pi_r)$ denote the number of $2k$-cycles in $Levi(\pi_r)$.

Explicit formulii for $c_{2k}(\pi_r)$ exists for

$k = 3, 4, 5, 6 - FL - K. Mellinger - O. Vega (09)$

$k = 7, 8, 9, 10 - A. Voropaev (13)$

In all these cases $c_{2k}(\pi_r)$ depend on $k$ and on $r$ ONLY!
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In all these cases $c_{2k}(\pi_r)$ depend on $k$ and on $r$ ONLY!

**QUESTION:** Is it true that there exists $k \geq 11$, such that

$$c_{2k}(\pi_q) \neq c_{2k}(\text{PG}(2, q)) ?$$
4. Hamiltonian cycles and weak pancyclicity

Let $f, g : \mathbb{N} \to (0, \infty)$. We write

\[ f = o_n(g) = o(g), \quad \text{if } f/g \to 0, \ n \to \infty. \]

Let $(n_i)$ be a sequence of positive integers, $n_i \to \infty$.

Let $\Gamma_i = (V_{n_i}, E_{n_i})$ – a sequence of simple graphs, $|V_{n_i}| = n_i$.

If

\[ |E_{n_i}| = o_i(n_i^2) \]

and say that $\Gamma_i$ forms a sequence of sparse graphs.

If $\Gamma_i$ is $d_i$-regular, $(\Gamma_i)$ sparse iff $d_i = o_i(n_i)$.

**Example.** $\text{Levi}(\pi_q)$ – Levi graph of a projective plane $\pi_q$ of order $q$ (bipartite point-line incidence graph of $\pi_q$):

Then $n_q = 2(q^2 + q + 1)$, $q$ is a prime power, $d_q = q + 1 = o_q(n_q)$: $\text{Levi}(\pi_q)$ is sparse.
Hamiltonian cycles and weak pancyclicity

Γ is **hamiltonian** if it contains a spanning cycle (≡ hamiltonian cycle).

**G. Dirac (1952):** Let Γ have \( n \) vertices, \( n \geq 3 \). If \( d_\Gamma(x) = d(x) \geq n/2 \) for every vertex \( x \) of Γ, then Γ is hamiltonian.

**O. Ore (1960):** Let Γ have \( n \) vertices, \( n \geq 3 \). If \( d(x) + d(y) \geq n \) for every pair of non-adjacent vertices \( x \) and \( y \), then Γ is hamiltonian.

**Closure** \( cl(\Gamma) \) is a graph obtained from Γ by repeatedly adding a new edge \( xy \), connecting a nonadjacent pair of vertices \( x \) and \( y \) such that \( d(x) + d(y) \geq n \).

**A. Bondy - V. Chvátal (1972):** Γ is hamiltonian iff \( cl(\Gamma) \) is hamiltonian.
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None of these theorems implies that \( Levi(π_q) \) is hamiltonian.
Hamiltonian cycles and weak pancyclicity

Is $\text{Levi}(\pi_q)$ is hamiltonian?

**J. Singer (1938):** Yes, if $\pi_q = PG(2, q)$ – the classical plane.

**E. Schmeichel (1989):** Yes, for $\pi_p = PG(2, p)$ (different cycle).
Hamiltonian cycles and weak pancyclicity

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**FL - K. Mellinger - O. Vega (2013):** Yes, for all $\pi_r$. Moreover, $Levi(\pi_r)$ contains a cycle of length $2k$ for every $3 \leq k \leq r^2 + r + 1$. (weakly pancyclic)

Levi($\pi_r$) is also known as:
- a $(r+1, 6)$-cage;
- a bipartite 4-cycle-free graph with partitions of size $r^2 + r + 1$ and having the maximum number of edges;
- a generalized 3-gon of order $r$: a bipartite $(r+1)$-regular graph of diameter 3 and girth 6.
Hamiltonian cycles and weak pancyclicity

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Hamiltonian cycles and weak pancyclicity

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Hamiltonian cycles and weak pancyclicity

A generalized d-gon of order $r$ is a geometry with Levi graph being a bipartite $(r + 1)$-regular graph of diameter $d$ and girth $2d$. 
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Existence for $r \geq 2$ and $d \geq 3$:

J. Tits (1959) for $d \in \{4, 6\}$ $r = q$ – any prime power.

W. Feit and G. Higman (1964): If $r \geq 2$ and $d \geq 3$, it can exists only for $d \in \{3, 4, 6\}$.

Here are equivalent definitions:

- a $(r + 1, 2d)$-cage;

- a bipartite graph of girth at least $2d$ with partitions of size $r^{d-1} + r^{d-2} + \cdots + r + 1$ and having the maximum number of edges.
Denote a generalized \((r + 1)\)-regular \(d\)-gon by \(\pi_r^d\), \(d = 3, 4, 6\).

\[\pi_r^3 = \pi_r - \text{projective plane of order } q.\]

Easy to show that \(|V(Levi(\pi_r^d))| = 2(r^{d-1} + r^{d-2} + \cdots + r + 1)|.\]

Hence, \(Levi(\pi_r^d)\) is sparse.

Is \(Levi(\pi_r^d)\), hamiltonian for \(d = 4, 6\)?

J. Alexander - FL - A. Thomason (2016+): Yes, provided \(r\) being sufficiently large.

Is \(Levi(\pi_r^d)\), weakly pancyclic for \(d = 4, 6\)?

Not known. J. Exoo confirmed the weak pancyclicity for \(Levi(\pi_3^4)\) and \(Levi(\pi_5^4)\).
More general questions:

Consider a graph $\Gamma$ on $n$ vertices with girth at least $2k + 1$ and having “MANY” edges.

Is $\Gamma$ hamiltonian? Is $G$ weakly pancyclic?
Hamiltonian cycles and weak pancyclicity

Let $G(n, p)$ be a random graph model, and $\Gamma \in G(n, p)$.

What is $\text{Prob}[\Gamma \text{ is hamiltonian}]$?

L. Posa (1972): If $\Gamma \in G(n, p)$ and $p = c \log n / n$, then $\Gamma$ is hamiltonian, i.e.,

$$\text{Prob}[\Gamma \text{ is hamiltonian}] \to 1, \ n \to \infty,$$

provided that constant $c$ is sufficiently large.

Note that the expected degree of a vertex of $\Gamma$ is

$$(n - 1)p \sim c \log n = o(n),$$

and the expected number of edges is $\sim (c/2)n \log n = o(n^2)$.

Hence, $\Gamma$ is sparse.
Pseudo-random graphs

Let $A(\Gamma)$ be the adjacency matrix of $\Gamma$. $A(\Gamma)$ is a real symmetric matrix, and so $A(\Gamma)$ has real eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. If $\Gamma$ is $d$-regular, then $\lambda_1 = d$. Let $\lambda = \lambda(\Gamma) := \max\{|\lambda_i|: i = 2, 3, \ldots, n\}$. The difference $d - \lambda$ is called the spectral gap.
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\]

Let \(\Gamma\) be a \(d\)-regular graph on \(n\) vertices.

The difference \(d - \lambda\) is called the spectral gap.
Hamiltonian cycles and weak pancyclicity

It turns out that the spectral gap $d - \lambda$ is responsible for the pseudo-random properties of graphs:

The larger the spectral gap is, the closer the edge distribution of $\Gamma$ approaches that of a random graph $\mathcal{G}(n, d/n)$.

It is known that if $d \leq (1 - \epsilon)n$ for some $\epsilon > 0$, then $\lambda \geq c\sqrt{d}$.

If

$$c_1\sqrt{d} < \lambda < c_2\sqrt{d}$$

for some $c_1, c_2 > 0$, then $\Gamma$ is a "good" pseudo-random graph.

For $\Gamma \in \mathcal{G}(n, 1/2)$,

$$\lambda(\Gamma) \approx 2\sqrt{n/2} \approx 2\sqrt{d}.$$
Hamiltonian cycles and weak pancyclicity

Theorem (M. Krivelevich, B. Sudakov (2002))

Let $\Gamma$ be a $d$-regular $n$-vertex graph. If $n$ is large enough and

$$\lambda \leq \frac{(\log \log n)^2}{1000 \log n \log \log \log n} d,$$

then $\Gamma$ is Hamiltonian.
Hamiltonian cycles and weak pancyclicity

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then $\Gamma$ is Hamiltonian.

If $\Gamma$ is bipartite, then $\lambda = | - d| = d$, and the condition fails.
E.g., it fails for $\text{Levi}(\pi_r^d)$. 
Theorem (J. Alexander - FL - A. Thomason (2016+))

Let $\Gamma$ be a $d$-regular $n$-vertex bipartite graph, and

$$\lambda = \lambda(\Gamma) := \max\{|\lambda_i| : i = 2, 3, \ldots, n - 1\}.$$ 

If $n$ is large enough and

$$\lambda \leq \frac{(\log \log n)^2}{2000 \log n (\log \log \log n)} d,$$

then $\Gamma$ is Hamiltonian.

This gave another motivation for determining bounds on $\lambda$. 
Hamiltonian cycles and weak pancyclicity

The following bipartite graphs and some of their subgraphs are hamiltonian when their order is sufficiently large:

- Generalized polygons $\pi^d_r$ and their biaffine parts

- S. Cioabă - FL - W. Li (2014): Wenger graphs $W_n(q)$


- E. Moorehouse - S. Sun - J. Williford (2017): graphs $D(4, q)$
Hamiltonian cycles and weak pancyclicity

Problem 1: Establish similar results for all $q$, not just sufficiently large.

Problem 2: Strengthen a result of A. Frieze - M. Krivelevich (02) on
the hamiltonicity of random subgraphs of $d$-regular pseudo-random graphs.

This allows to show existence of smaller cycles in the graph.

At this time it is useful only for $d \gg n^{3/4} (\log n)^3$ and large $n$.

Thank you!