ALGEBRAIC DIGRAPHS

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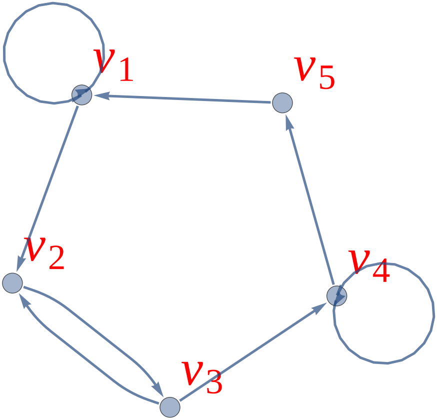
What is a digraph?



Definition

A *digraph* is a pair $D = (V, A)$ of:

- a set V, whose elements are called *vertices* or *nodes*
- a set A of ordered pairs of vertices, called *arcs*, *directed edges*, or *arrows*

What is an algebraic digraph $D(q; \mathbf{f})$?

Let

$\mathbb{F}_q$ be a *finite field* with q elements;

$\mathbf{f} = (f_1, \ldots, f_l)\colon \mathbb{F}_q^2 \to \mathbb{F}_q^l$, each f_i is a bivariate polynomial (*defining polynomials*).

Definition

An *algebraic digraph*, denoted $D(q; \mathbf{f})$, is a digraph whose

vertex set is $\mathbb{F}_q^{l+1}$

arc set consits of ordered pairs $\big((x_1, \ldots, x_{l+1}), (y_1, \ldots, y_{l+1})\big)$ with the relation

$$x_i + y_i = f_{i-1}(x_1, y_1), \quad \text{for all } i,\ 2 \le i \le l+1.$$

Example of $D(q; \mathbf{f})$

Example of $D(q; \mathbf{f})$

$V(D) = \mathbb{F}_q^4$

$\mathbf{f} = (f_1, f_2, f_3) \colon \mathbb{F}_q^2 \to \mathbb{F}_q^3$

There is an arc from vertex (x_1, x_2, x_3, x_4) to vertex (y_1, y_2, y_3, y_4) if and only if

$$\begin{aligned} x_2 + y_2 &= x_1 y_1^3 + 3x_1^2 y_1^5 &= f_1(x_1, y_1) \\ x_3 + y_3 &= 2x_1 y_1^2 &= f_2(x_1, y_1) \\ x_4 + y_4 &= x_1 y_1 + x_1^2 y_1^3 &= f_3(x_1, y_1) \end{aligned}$$

What is a monomial algebraic digraph?

Definition

A *monomial algebraic digraph*, denoted $D(q; m, n)$, is an algebraic graph in which

- vertex set V is $\mathbb{F}_q^2$, so $\mathbf{f} = f_1$, and
- there is an arc from (x_1, x_2) to (y_1, y_2) if and only if

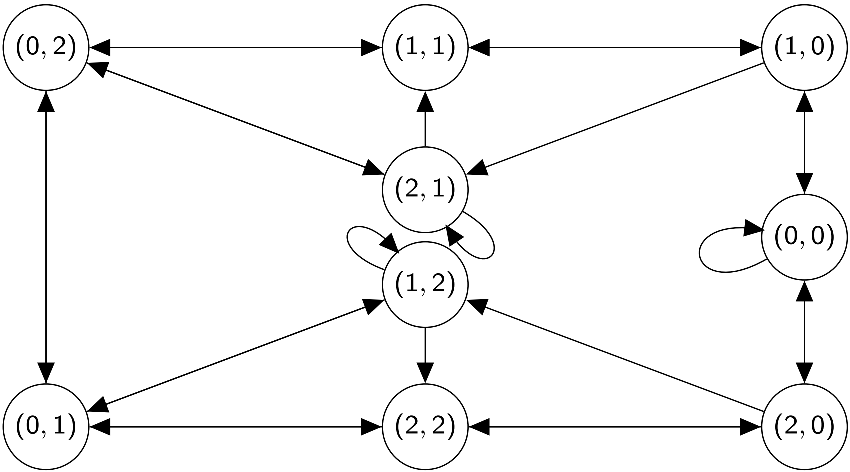
$$x_2 + y_2 = x_1^m y_1^n =: f_1(x_1, y_1).$$

$D(3; 1, 2)$

- has $3^2 = 9$ vertices, each is a pair (x, y), $x, y \in \mathbb{F}_3$
- there is $(x_1, x_2) \to (y_1, y_2)$ if and only if

$$x_2 + y_2 = x_1^1 y_1^2$$

$D(3;1,2)$: $x_2 + y_2 = x_1^1 y_1^2$



Motivation (from Extremal Graph Theory)

Work of:

Lazebnik, Woldar (2001)

Viglione (2001)

Dmytrenko, Lazebnik, Viglione (2005)

Bipartite udirected $B\Gamma_n$

$V(B\Gamma_n) = P_n \cup L_n$, both P_n and L_n are copies of $\mathbb{F}_q^n$

point $(p) = (p_1, \dots, p_n)$ is adjacent to line $[l] = (l_1, \dots, l_n)$ if

$$\begin{aligned} l_2 + p_2 &= f_2(p_1, l_1) \\ l_3 + p_3 &= f_3(p_1, l_1, p_2, l_2) \\ &\vdots \\ l_n + p_n &= f_n(p_1, l_1, p_2, l_2, \dots, p_{n-1}, l_{n-1}). \end{aligned}$$

Application of $B\Gamma_n$ to extremal problems

Let

C_n denote the cycle of length $n \geq 3$

$ex(v, \{C_3, C_4, \dots, C_{2k}\})$ denote the greatest number of edges in a graph or order v which contains no subgraphs isomorphic to any $C_3, \dots, C_{2k}$.

Theorem (Lazebnik, Ustimenko, Woldar 1995)

$$ex(v, \{C_3, C_4, \dots, C_{2k}\}) \geq c_k v^{1+\frac{2}{3k-3+\epsilon}},$$

where c_k is a positive function of k, and $\epsilon = 0$ if $k \neq 5$ is odd, and $\epsilon = 1$ if k is even.

This lower bounds comes from $B\Gamma_n$ with a certain choice of defining functions f_i, $2 \leq i \leq n$.

Some Upper Bounds on $ex(v, \{C_3, C_4, \ldots, C_{2k}\})$

Erdős, Bondy–Simonovits, 1974

$ex(v, \{C_3, C_4, \ldots, C_{2k}\}) \leq 20kv^{1+(1/k)}$, for v sufficiently large.

Verstraëte, 2000

$ex(v, \{C_3, C_4, \ldots, C_{2k}\}) \leq 8(k-1)v^{1+(1/k)}$, for v sufficiently large.

Pikhurko, 2012

$ex(v, \{C_3, C_4, \ldots, C_{2k}\}) \leq (k-1)v^{1+(1/k)} + O(v)$.

Bukh, Jiang, 2014

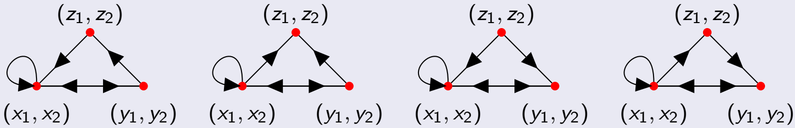
$ex(v, \{C_3, C_4, \ldots, C_{2k}\}) \leq 80\sqrt{k \log k} \cdot v^{1+(1/k)} + O(v)$.

A peculiar detail

Equal number of cycles

$D(17; 1, 4)$ and $D(17; 1, 12)$ are easy to show to be non-isomorphic, but they contain equal numbers of copies of C_k, for $3 \le k \le 13$.

...and other small digraphs (many of them)



Problems/Questions Addressed

✓ ~~(Strong) connectivity for the general digraph $D(q;\mathbf{f})$~~ (number and size of components as well as their structure)

- Diameter for monomial digraphs $\mathbf{f}(x,y) = f_1(x,y) = x^m y^n$ (techniques include Additive Combinatorics such as Waring's Problem; and Hasse-Weil bound). Most known bounds are not tight.
- Isomorphism problem: when are $D(q;\mathbf{f})$ and $D(q;\mathbf{h})$ isomorphic? We only have a partial result for monomial digraphs.

Diameter of $D(q; m, n)$

Theorem (Kodess, Lazebnik, Smith, Sporre, 2016)

If $D_q = D(q; m, n)$ is strong, then:

$$\mathrm{diam}(D_q(q; m, n)) \leq C\sqrt{\max\{m, n\} + 1}$$

Theorem (Kodess, Lazebnik, Smith, Sporre, 2016)

If $D_q = D(q; m, n)$ is strong, then:

if $q > (\max(m, n))^2$, *then* $\mathrm{diam}(D_q) \leq 49$

if $q > (\min(m, n))^4$, *then* $\mathrm{diam}(D_q) \leq 13$

Comment

Proofs are based on bounds on Waring's number by J. Cipra and T. Cochrane

Open Conjecture

Conjecture

$D(q; m_1, n_1) \cong D(q; m_2, n_2)$ if and only if there exists k coprime with $q-1$ such that

$$m_2 \equiv km_1 \mod (q-1),$$

$$n_2 \equiv kn_1 \mod (q-1).$$

Example

$D(13^7; 1, 2) \cong D(13^7; 5, 10)$

Comments

- easy to prove sufficiency
- partial results about necessity
- checked with `SageMath` for many different values of q

THANK YOU!