Median Eigenvalues and Graph Inverse

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Graphs and Matrices

Let G be a graph with edge set $E(G)$ and vertex set $V(G)$. The adjacency matrix of G is defined as

$$\mathbb{A} = [a_{ij}]$$

where $a_{ij} = 1$ if $ij \in E(G)$ and $a_{ij} = 0$ otherwise.

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where $a_{ij} = 1$ if $ij \in E(G)$ and $a_{ij} = 0$ otherwise.

The eigenvalues of $\mathbb{A}$ are also called the eigenvalues of the graph G, denoted by $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$.

Median Eigenvalues

Two median eigenvalues are denoted by $\lambda_H(G)$ and $\lambda_L(G)$, where $H = \lfloor \frac{n+1}{2} \rfloor$ and $L = \lceil \frac{n+1}{2} \rceil$.

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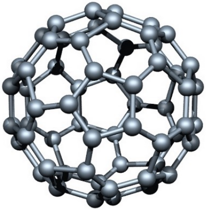

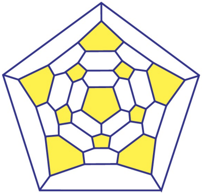

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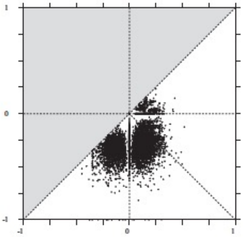

HOMO-LUMO map for the 7595 20-vertex cubic polyhedra.

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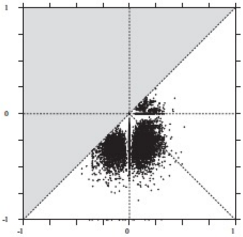

HOMO-LUMO map for the 7595 20-vertex cubic polyhedra.

- Conjecture (Fowler & Pisanski, 2010) There are finitely many subcubic graphs with $\lambda_H, \lambda_L \notin [-1, 1]$.

Bounding Median Eigenvalues: Graph Partition

- Interlacing Theorem: Let $\mathbb{A}$ be an $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Let B be an $(n-k) \times (n-k)$ symmetric minor of $\mathbb{A}$ with eigenvalues $\mu_1 \geq \cdots \geq \mu_{n-k}$. Then $\lambda_i \leq \mu_i \leq \lambda_{i+k}$.

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- Graph Partition: A partition $\{V_1, V_2\}$ of $V(G)$ is unfriendly if every vertex in V_1 has at least as many neigbhors in V_2 as in V_1. A partition is unbalanced if $|V_1| \neq |V_2|$.

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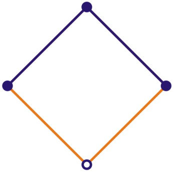

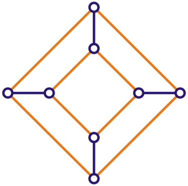

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- (Mohar, 2016) Every connected bipartite subcubic graph different from the Heawood graph has $\lambda_H, \lambda_L \in [-1, 1]$.

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- (MOHAR, 2016) Every connected bipartite subcubic graph different from the Heawood graph has $\lambda_H, \lambda_L \in [-1, 1]$.
- CONJECTURE (MOHAR, 2012) Every plane subcubic graph has $\lambda_H, \lambda_L \in [-1, 1]$.

Bounding Median Eigenvalues: Graph Partition

- (Y., 2017+) Let G be a 2-connected plane cubic graph different from K_4. Then G has an unfriendly unbalanced partition if and only if G has no 4-cycle 2-factor.

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- (Y., 2017+) Let G be a 2-connected plane cubic graph without a 4-cycle 2-factor. Then $\lambda_H(G), \lambda_L(G) \in [-1, 1]$.

Bounding Median Eigenvalues: Graph Square

- The square of a graph G is another graph with the same vertex set as G and two vertices are adjacent if they have distance two in G, denoted by G^2.

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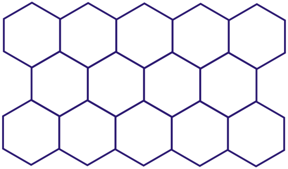

Adjacency matrix A

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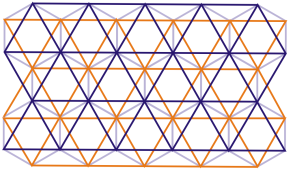

Adjacency matrix A^2

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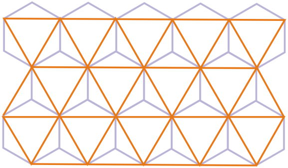
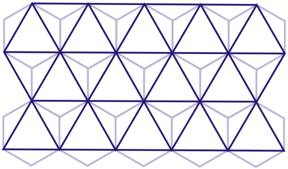


Adjacency matrix B_*



Adjacency matrix B_o

Bounding Median Eigenvalues: Graph Square

Let G be a bipartite graph with adjacency matrix $A = \begin{bmatrix} 0 & B \\ B^{\intercal} & 0 \end{bmatrix}$. Denote $B_* = BB^{\intercal}$ and $B_o = B^{\intercal}B$.

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- $A^2 = \begin{bmatrix} B_* & 0 \\ 0 & B_o \end{bmatrix}$

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- $A^2 = \begin{bmatrix} B_* & 0 \\ 0 & B_o \end{bmatrix}$
- $\max\{|\lambda_L|, |\lambda_H|\} \leq \min\{\sqrt{\lambda_{\min}(B_*)}, \sqrt{\lambda_{\min}(B_o)}\}$

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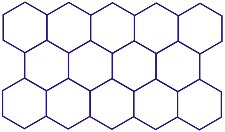

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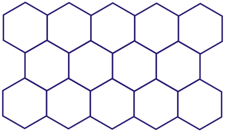
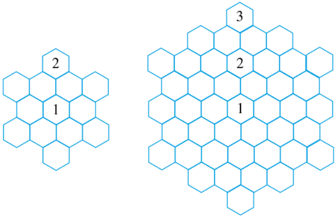


- (Klein, Yang and Y., 2015) A hexagonal system has median eigenvalues

$$-\sqrt{\frac{n_2}{n}} \leq \lambda_L \leq \lambda_H \leq \sqrt{\frac{n_2}{n}}.$$

Bounding Median Eigenvalues: Graph Square

- Ring-type hexagonal systems: $\Delta \leq \sqrt{(8x-4)/(3x^2-3x+1)}$ where x is the number of rings.



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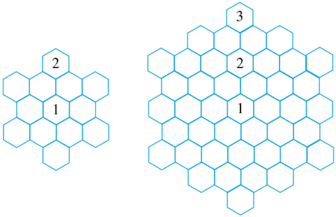
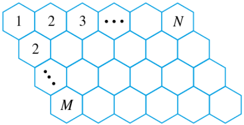


- Parallelogram-like hexagonal systems: $\Delta \leq 2\sqrt{(M+N+1)/(MN+M+N)}$.



Bounding Median Eigenvalues: Graph Inverse

- Rayleigh-Quotient method:

$$\lambda_1(G) = \max_{\mathbf{x} \in \mathbb{R}^n} \frac{<\mathbf{x}, \mathbb{A}\mathbf{x}>}{<\mathbf{x}, \mathbf{x}>};$$

$$\lambda_n(G) = \min_{\mathbf{x} \in \mathbb{R}^n} \frac{<\mathbf{x}, \mathbb{A}\mathbf{x}>}{<\mathbf{x}, \mathbf{x}>}.$$

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- The spectrum of a graph G is symmetric about the origin if and only if G is bipartite graph.

  If $\mathbb{A}$ is invertible, then $\lambda_H(G) = 1/\lambda_1(\mathbb{A}^{-1})$ and $\lambda_L(G) = 1/\lambda_n(\mathbb{A}^{-1})$.

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- For non-bipartite graph, the method works as long as the spectrum splits about the origin (i.e., half number of eigenvalues are positive and half of them are negative), such as leapfrog fullerenes.

Graph Inverse

- (Harary, 1962) Let G be a connected graph and $\mathbb{A}$, the adjacency matrix of G. Then $\mathbb{A}$ is invertible and its inverse is an adjacency matrix of a graph if and only if G is K_2.

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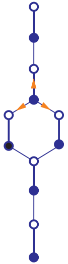
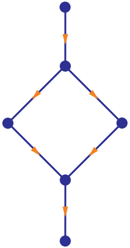
- (GODSIL, 1986) Define inverse for bipartite graphs G with unique perfect matching.

  A bipartite graph G is *invertible* if $\mathbb{A}$ is invertible and diagonally similar to a non-negative matrix (i.e. there exists a diagonal matrix $\mathbb{D}$ with entry in $\{0, 1, -1\}$ such that $\mathbb{D}\mathbb{A}^{-1}\mathbb{D}$ is an adjacency matrix of a graph).

Graph Inverse

- (Simon & Cao, 1989) Let G be a bipartite graph with a bipartition (R, C). Let D be the digraph obtained from G by orienting edges from R to C and then contracting all edges in M. Then D is acyclic.





Graph Inverse

- For the digraph D, define a poset $(\mathcal{P}, \leq)$ such that $\mathcal{P} = V(D)$ and $a_i \leq a_j$ if and only if there is a directed path from a_i to a_j,

$$(\mathbb{M}(x))_{ij} := \begin{cases} 1 & \text{if } a_i a_j \text{ is an arc of } D; \\ x & \text{if } a_i a_j \notin E(D) \text{ and } a_i \leq a_j \text{ in } \mathcal{P}; \\ 0 & \text{otherwise.} \end{cases}$$

- $\mathbb{M}(1)$ is the Zeta-function matrix of $\mathcal{P}$ and $\mathbb{M}(0) = \mathbb{B}$ is the adjacency matrix of D.

- (Lovász, 1979) $\mathbb{M}^{-1}(1)$ is the Möbius-function matrix of $\mathcal{P}$ and $\mathbb{M}^{-1}(0)$ is the inverse of bipartite adjacency matrix of G .

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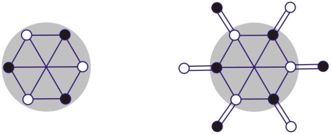
- (SIMION & CAO, 1989) Every bipartite corona graph G is invertible.



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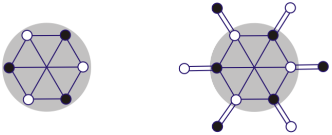

- (AKBARI & KIRKLAND, 2007) Characterize all invertible unicycle graphs.

Graph Inverse

- DEFINITION. Let (G, ω) be a weighted graph with $\omega : E(G) \to \mathbb{F} \backslash \{0\}$ where $\mathbb{F}$ is a feild. Let $\mathbb{A}$ be the adjacency matrix of (G, ω) defined by $\mathbb{A}_{i,j} = \omega(ij)$ if $ij \in E(G)$ and 0 otherwise.

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- If $\mathbb{A}$ is invertible, then its inverse is symmetric and hence is an adjacency matrix of a weighted graph, denoted by (G^{-1}, ω^{-1}).

Graph Inverse

A spanning subgraph S is a 2-matching if every component of G is either K_2 or a cycle (including loop). Let $S = \mathcal{C} \cup M \cup L$ where $\mathcal{C} = \{\text{all cycles of } S\}$, $L = \{\text{all loops of } S\}$ and $M = \{\text{all } K_2 \text{ of } S\}$.

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- (Harary 1967) Let (G, ω) be a weighted graph and $\mathbb{A}$ the adjacency matrix. Then

$$\det(\mathbb{A}) = \sum_{S} 2^{|\mathcal{C}|}\omega(\mathcal{C} \cup L)\ \omega^2(M)(-1)^{|\mathcal{C}|+|L|+|E(S)|},$$

where $S = C \cup M \cup L$ is a 2-matching.

Graph Inverse

- (Klein, Mandel, Yang & Y., 2017) Let (G, ω) be a weighted graph with adjacency matrix $\mathbb{A}$, and

$$\mathcal{P}_{ij} = \{P | P \text{ joins } i \text{ and } j \text{ such that } G - V(P) \text{ has a 2-matching } S\}.$$

If (G, ω) has an inverse (G^{-1}, ω^{-1}), then

$$\omega^{-1}(ij) = \frac{1}{\det(\mathbb{A})} \sum_{P \in \mathcal{P}_{ij}} \Big(\omega(P) \big(\sum_{S} \omega(\mathcal{C} \cup L)\ \omega^2(M)\ 2^{|\mathcal{C}|} (-1)^{|\mathcal{C}|+|L|+|E(S) \cup E(P)|} \big) \Big)$$

where $S = \mathcal{C} \cup M \cup L$ is a 2-matching of $G - V(P)$.

Graph Inverse

- Let G be a bipartite graph with a unique perfect matching M. Then G has an inverse (G^{-1}, w^{-1}) such that

$$w^{-1}(ij) = \begin{cases} \sum_P (-1)^{|E(P)\setminus M|} & \text{if } i \neq j; \\ 0 & \text{if } i = j. \end{cases}$$

where P is an M-alternating path of G.

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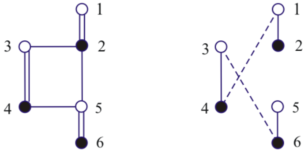

Graph Inverse

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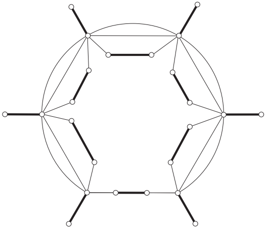
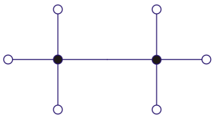


Bounding Median Eigenvalues



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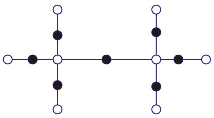
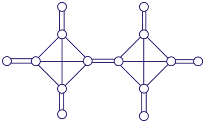


sub(T)



st(T)[1]

[1] J.J. Sylvester, Am. J. Math. 1 (1878) 64-104.

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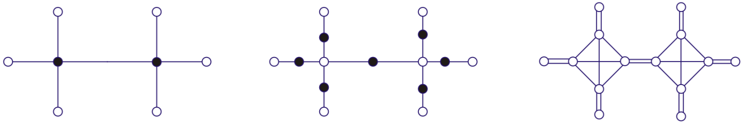

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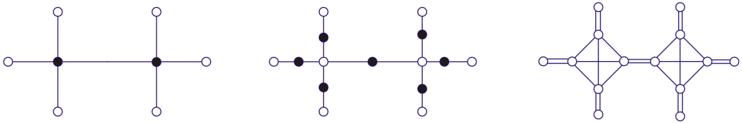

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- The median eigenvalues of a stellated tree satisfy $-1 \leq \lambda_L < 0 < \lambda_H \leq 1$.

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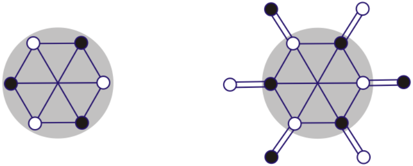
Bounding Median Eigenvalues



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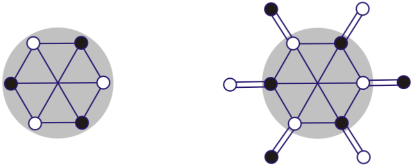

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Bounding Median Eigenvalues

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Bounding Median Eigenvalues

- (Pisanski & Fowler, 2010). Every tree with maximum degree at most 3 has median eigenvalues in $[-1, 1]$.
- (Wu, Yang & Y., 2017+). Every bipartite graph with at most one perfect matching has median eigenvalues in $[-1, 1]$.

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- PROBLEM: Which graphs have split spectrums?

THANK YOU!