Partially metric association schemes with a small multiplicity

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Joint work with Jack Koolen and Jongyook Park
Celebrating the work of....

Willem Haemers, Felix Lazebnik, and Andrew Woldar
Overview and intro

Association schemes are symmetric (except at the end of the talk)
We interpret relations of schemes as graphs (scheme graphs)
Every association scheme has an extremely small multiplicity (1)
Product constructions ? Multiplicity 2 ?
Focus on multiplicity 3. Which schemes are most interesting?
Partially metric schemes!
Tools: cosines, Godsil’s bound, Terwilliger’s light tail, Yamazaki’s lemma
All partially metric schemes with a multiplicity 3
Time? Nonsymmetric schemes?
Let $X$ be a finite set. An *association scheme* with rank $d + 1$ on $X$ is a pair $(X, \mathcal{R})$ such that

(i) $\mathcal{R} = \{R_0, R_1, \cdots, R_d\}$ is a partition of $X \times X$,

(ii) $R_0 := \{(x, x) \mid x \in X\}$,

(iii) $R_i = R_i^\top$ for each $i$, i.e., if $(x, y) \in R_i$ then $(y, x) \in R_i$,

(iv) there are numbers $p^h_{ij}$ — the *intersection numbers* of $(X, \mathcal{R})$ — such that for every pair $(x, y) \in R_h$ the number of $z \in X$ with $(x, z) \in R_i$ and $(z, y) \in R_j$ equals $p^h_{ij}$. 

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Algebra

(i)' \[ \sum_{i=0}^{d} A_i = J, \text{ where } J \text{ is the all-one matrix}, \]

(ii)' \[ A_0 = I, \text{ where } I \text{ is the identity matrix}, \]

(iii)' \[ A_i^\top = A_i \text{ for all } i, \]

(iv)' \[ A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h. \]

The Bose-Mesner algebra \( \mathcal{M} = \langle A_i \mid i = 0, \ldots, d \rangle \) has a basis of minimal scheme idempotents \( E_0 = \frac{1}{n} J, E_1, \ldots, E_d \). The rank of \( E_j \) is denoted by \( m_j \) and is called the multiplicity of \( E_j \), for \( 0 \leq j \leq d \).

\[ A_i = \sum_{j=0}^{d} P_{ji} E_j \quad \text{and} \quad E_j = \frac{1}{n} \sum_{i=0}^{d} Q_{ij} A_i. \]
Geometry

\[(E_j)_{xx} = \frac{Q_{ij}}{n} = \frac{m_j}{n} \text{ for all } x \in X.\]

For \((x, y) \in R_i\), let \(\omega_{xy} = \omega_{xy}(j) = \frac{(E_j)_{xy}}{(E_j)_{xx}} = \frac{Q_{ij}}{m_j}\). We call these numbers \(\omega_i = \omega_i(j) = \frac{Q_{ij}}{m_j}\) the cosines corresponding to \(E_j\), and note that \(\omega_0 = 1\).

If \(E\) is a minimal idempotent with multiplicity \(m\), then \(E = UU^\top\), with \(U\) an \(n \times m\) matrix with columns forming an orthonormal basis of the eigenspace of \(E\) for its eigenvalue 1.

For every vertex \(x\) we denote by \(\hat{x}\) the row of \(U\) that corresponds to \(x\), normalized to length 1. Now the inner product \(\langle \hat{x}, \hat{y} \rangle\) is equal to \(\frac{n}{m}E_{xy} = \omega_{xy}\).
Interesting schemes

A scheme is called *primitive* if all nontrivial relations are connected.

**Bannai and Bannai 2006**

The only primitive scheme with a multiplicity 3 is the scheme of the tetrahedron ($K_4$).
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The *direct product* of $(X, \mathcal{R})$ and $(X', \mathcal{R}')$ is the association scheme with relation matrices $A_i \otimes A'_j$ for $i = 0, 1, \ldots, d$ and $j = 0, 1, \ldots, d'$.

Starting from an association scheme $(X, \mathcal{R})$ with a multiplicity 3, one can construct other association schemes with a multiplicity 3 by taking the direct product of $(X, \mathcal{R})$ with any other scheme. Also other kinds of product constructions for association schemes are possible, giving rise to many association schemes with a multiplicity 3.
Distance-regular graphs

For a connected graph $\Gamma$ with diameter $D$, the distance-$i$ graph $\Gamma_i$ of $\Gamma$ ($0 \leq i \leq D$) is the graph whose vertices are those of $\Gamma$ and whose edges are the pairs of vertices at mutual distance $i$ in $\Gamma$. A connected graph is called distance-regular if the distance-$i$ graphs ($0 \leq i \leq D$) form an association scheme (a so-called metric scheme).
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Godsil 80s

The distance-regular graphs with a multiplicity 3 are the graphs of the Platonic solids and the regular complete 4-partite graphs.

Distance-regular graphs with a multiplicity up to 8 have been classified (Zhu, Martin, Koolen, Godsil 90s).
Partially metric schemes

Yamazaki 1998

If $\Gamma$ is a connected cubic scheme graph, then the distance-2 graph is also a relation of the scheme.

We call a scheme \textit{partially metric} if it has a connected scheme graph whose distance-2 graph is also a relation of the scheme.

We adopt ‘distance-regular graph’ notation as far as possible: $a_1 = p_{11}^1$, etc.
Partially metric schemes with a multiplicity 3

Let \((X, R)\) be a partially metric scheme with rank \(d + 1\) and a multiplicity 3, and let \(\Gamma\) be the corresponding scheme graph. Then

(i) \(d = 1\) and \(\Gamma\) is the tetrahedron (the complete graph on 4 vertices),
(ii) \(d = 3\) and \(\Gamma\) is the cube,
(iii) \(d = 5\) and \(\Gamma\) is the Möbius-Kantor graph,
(iv) \(d = 6\) and \(\Gamma\) is the Nauru graph,
(v) \(d = 11\) and \(\Gamma\) is the Foster graph F048A,
(vi) \(d = 5\) and \(\Gamma\) is the dodecahedron,
(vii) \(d = 11\) and \(\Gamma\) is the bipartite double of the dodecahedron,
(viii) \(d = 3\) and \(\Gamma\) is the icosahedron,
(ix) \(d = 2\) and \(\Gamma\) is the octahedron,
(x) \(d = 2\) and \(\Gamma\) is a regular complete 4-partite graph.

Moreover, \((X, R)\) is uniquely determined by \(\Gamma\). In all cases, except (vii), this is the scheme that is generated by \(\Gamma\). In case (vii), the scheme is the bipartite double scheme of the scheme of case (vi).
Godsil’s bound

**Godsil’s bound (Cámara et al. 2013)**

Let \((X, \mathcal{R})\) be a partially metric association scheme and assume that the corresponding scheme graph \(\Gamma\) has valency \(k \geq 3\). Let \(E\) be a minimal scheme idempotent of \((X, \mathcal{R})\) with multiplicity \(m\) for corresponding eigenvalue \(\theta \neq \pm k\). If \(\Gamma\) is not complete multipartite, then

\[
k \leq \frac{(m + 2)(m - 1)}{2}.
\]

\(m = 1, 2\) are trivial.

\(m = 3\) and complete multipartite implies regular complete 4-partite or the octahedron (cases (ix) and (x)).

\(m = 3, k \leq 5\) ?
\[ m = 3, \ k > 3 \]

\[
k > m \text{ implies } p_{11}^1 = a_1 > 0 \text{ because of local eigenvalues (à la Terwilliger).}
\]

\[
k = 5, \ a_1 = 2 \text{ gives the icosahedron (case (viii)).}
\]

\[
k = 4, \ a_1 = 2 \text{ gives the octahedron (case (ix)).}
\]

\[
k = 4, \ a_1 = 1 \text{ implies } (m = 3)-\text{eigenvalue } \theta = -2, \text{ which is excluded because of a light tail argument (à la Jurišić, Terwilliger, Žitnik).}
\]
$m = 3$, cubic graphs

$k = 3, a_1 > 0$ gives the tetrahedron ($K_4$; case (i)).

$k = 3, a_1 = 0, c_2 > 1$ gives the cube (case (ii)).

What remains: $k = 3, a_1 = 0, c_2 = 1$.

$\Gamma$ is bipartite. If not, then consider bipartite double scheme (direct product with $K_2$ scheme).
Tools: cosines

\[ \theta E = AE \text{ implies } \theta \omega_h = \sum_{\ell=1}^{d} p_{1\ell}^h \omega_{\ell} \]

- \( \omega_0 = 1 \),
- \( \theta \omega_0 = 3\omega_1 \), so \( \omega_1 = \theta/3 \),
- \( \theta \omega_1 = \omega_0 + 2\omega_2 \), so \( \omega_2 = \frac{1}{6}(\theta^2 - 3) \).
Tools: Yamazaki’s lemma

Yamazaki 1998

\[ c_{i+1}(x, z_3) = 1 \]

\[ i \geq 2 \]
Let $u_1$ and $u_2$ be two adjacent vertices in $\Gamma$, let $v_1, v_2$ be the other two neighbors of $u_1$, and $v_3, v_4$ be the other two neighbors of $u_2$. Fix another vertex $x$, and let $\psi_i = \omega_{xu_i}$ ($i = 1, 2$) and $\phi_i = \omega_{ xv_i}$ ($i = 1, 2, 3, 4$) be the respective cosines corresponding to $E$. Then

$$\phi_3, \phi_4 = \frac{1}{2}(\theta \psi_2 - \psi_1 \pm (\phi_1 - \phi_2)).$$
The relevant eigenvalues

\begin{itemize}
  \item \( \omega_0 = 1, \omega_1 = \frac{1}{3} \theta, \omega_2 = \frac{1}{6}(\theta^2 - 3) \)
  \item \( \omega_3,4 = \frac{1}{2}(\theta \omega_2 - \omega_1 \pm \omega_0 \mp \omega_2) = \frac{1}{12}(\theta^3 \mp \theta^2 - 5\theta \pm 9) \)
  \item \( \omega_5 = \frac{1}{2}(\theta \omega_3 - \omega_2 + \omega_1 - \omega_4) = \frac{1}{24}(\theta^4 - 2\theta^3 - 8\theta^2 + 18\theta + 15) \)
  \item \( \omega_6 = \frac{1}{2}(\theta \omega_3 - \omega_2 - \omega_1 + \omega_4) = \frac{1}{24}(\theta^4 - 6\theta^2 - 3) \)
  \item \( \omega_7 = \frac{1}{2}(\theta \omega_4 - \omega_2 + \omega_1 - \omega_3) = \frac{1}{24}(\theta^4 - 6\theta^2 - 3) \)
  \item \( \omega_8 = \frac{1}{2}(\theta \omega_4 - \omega_2 - \omega_1 + \omega_3) = \frac{1}{24}(\theta^4 + 2\theta^3 - 8\theta^2 - 18\theta + 15) \)
\end{itemize}

\( \omega_i \neq \omega_2 \) for \( i = 5, 6, 7, 8 \), hence \( c_3 = 1 \) if \( \theta \neq \pm 1 \)
$c_3 = 1$

Yamazaki: $p_{14}^5 \neq 0$ or $p_{14}^6 \neq 0$

$\omega_9$ and the double fork implies that $p_{14}^5 = 0$ and

$\theta = \pm 1, \pm \sqrt{5}$
The dodecahedron and its bipartite double

\[
\begin{align*}
\omega_0 &= 1 \\
\omega_1 &= \frac{\sqrt{5}}{3} \\
\omega_2 &= \frac{1}{3} \\
\omega_3 &= \frac{1}{3} \\
\omega_4 &= -\frac{1}{3} \\
\omega_5 &= \frac{\sqrt{5}}{3} \\
\omega_6 &= \omega_7 = -\frac{1}{3} \\
\omega_8 &= -\frac{\sqrt{5}}{3} \\
\omega_9 &= -\frac{\sqrt{5}}{3} \\
\omega_{10} &= -1 \\
\omega_{11} &= 1 \\
\omega_{12} &= -1
\end{align*}
\]
The dodecahedron has spectrum \( \{3^1, \sqrt{5}^3, 1^5, 0^4, -2^4, -\sqrt{5}^3\} \).
Bipartite double has spectrum \( \{3^1, \sqrt{5}^6, 2^4, 1^5, 0^8, -1^5, -2^4, -\sqrt{5}^6, -3^1\} \).
Partial relation-distribution diagram for $\theta = 1$
The Möbius-Kantor graph

The Möbius-Kantor graph is the unique double cover of the cube without 4-cycles. It is isomorphic to the generalized Petersen graph $GP(8,3)$ and has spectrum \{3, $\sqrt{3}4$, 13, $-13$, $-\sqrt{3}4$, $-31$\}. It is 2-arc-transitive and also known as the Foster graph F016A.
The Möbius-Kantor graph is the unique double cover of the cube without 4-cycles. It is isomorphic to the generalized Petersen graph $GP(8, 3)$ and has spectrum $\{3^1, \sqrt{3}^4, 1^3, -1^3, -\sqrt{3}^4, -3^1\}$. It is 2-arc-transitive and also known as the Foster graph F016A.
The Nauru graph is a triple cover of the cube. It is isomorphic to the generalized Petersen graph \( GP(12, 5) \) and has spectrum \( \{3, 2, 6, 1, 0, 4, -1, -2, -3\} \). It is 2-arc-transitive and also known as the Foster graph \( F^{024A} \).
The Nauru graph

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Girth 8
The Foster graph F048A is a 6-cover of the cube, a 3-cover of the Möbius-Kantor graph, and a 2-cover of the Nauru graph. It is isomorphic to the generalized Petersen graph $GP(24, 5)$ and has spectrum $\{3, \sqrt{6}, 2, \sqrt{3}, 1, 0, -\frac{1}{3}, -\sqrt{3}, -2, -\sqrt{6}, -3\}$.
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\{3^1, \sqrt{6}^4, 2^6, \sqrt{3}^4, 1^3, 0^{12}, -1^3, -\sqrt{3}^4, -2^6, -\sqrt{6}^4, -3^1\}.
\]
The Coxeter graph

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There are no such association schemes
Another 3-cover of the Möbius-Kantor graph

There is no such association scheme
Another 3-cover of the Möbius-Kantor graph

There is no such association scheme

Let $C$ be an order $n$ cyclic permutation matrix and $N = \begin{bmatrix}
I & I & I & 0 \\
I & C & 0 & I \\
I & 0 & C^{k+1} & C^k \\
0 & I & C^k & C^k
\end{bmatrix}$.

Let $n$ and $k \leq n - 1$ be such that $k^2 + k + 1$ is a multiple of $n$. Then the bipartite graph $\Gamma$ with bipartite incidence matrix $N$ is arc-transitive and it has eigenvalues $\pm 1$ with multiplicity three.
Non-symmetric schemes


Let $C$ be an order $n$ cyclic permutation matrix and $N = \begin{bmatrix}
I & I & I & 0 \\
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$(n, k) = (1, 0)$ and $(n, k) = (3, 1)$: cube and Nauru graph.
All other examples: non-symmetric (non-commutative) schemes.

An infinite family of non-commutative association schemes with a connected symmetric relation having an eigenvalue with multiplicity 3.
Recess