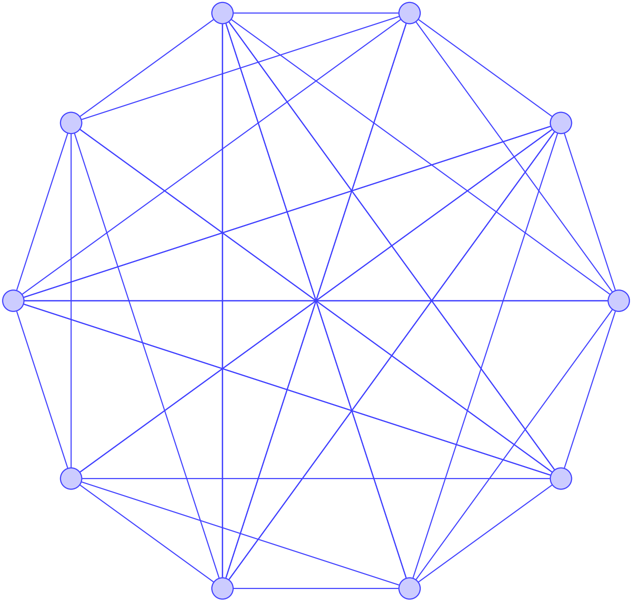
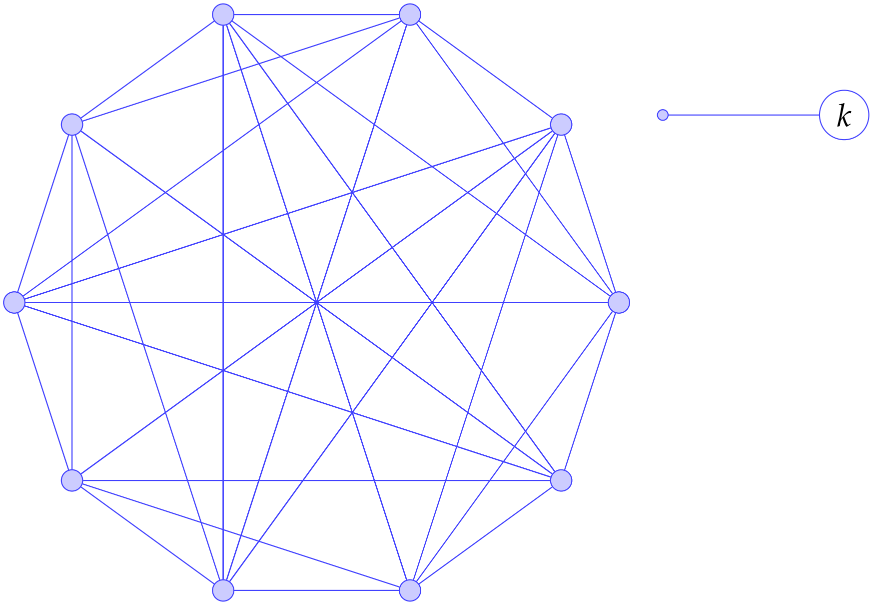
Edge-regular graphs and regular cliques

Gary Greaves

NTU, Singapore

7th August 2017

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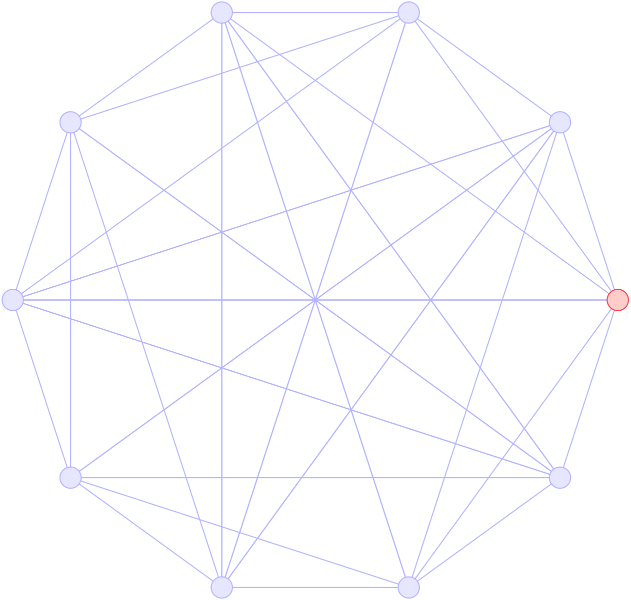

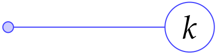

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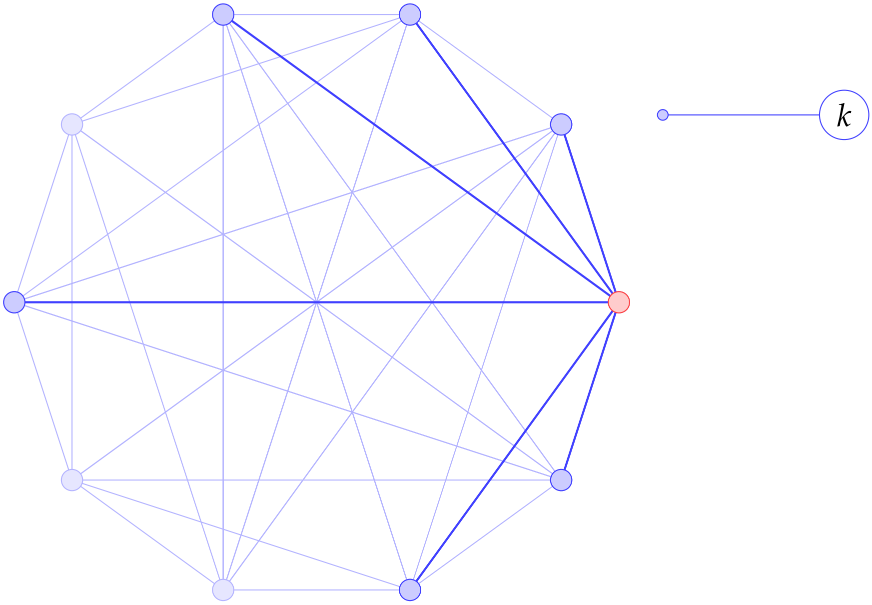
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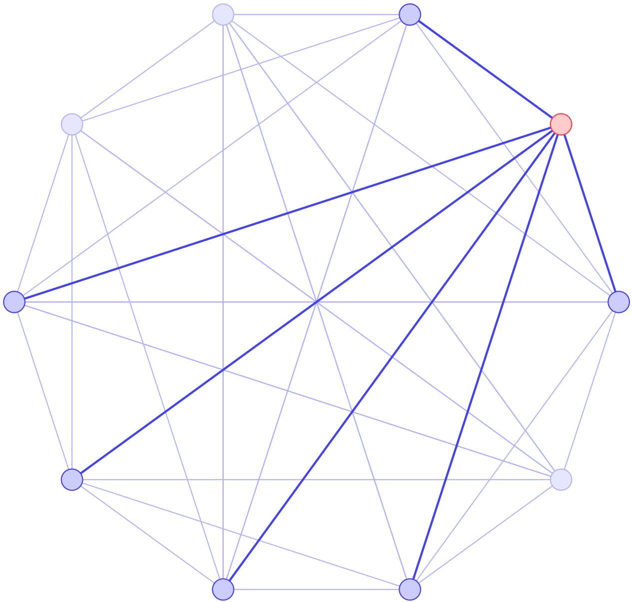

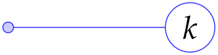
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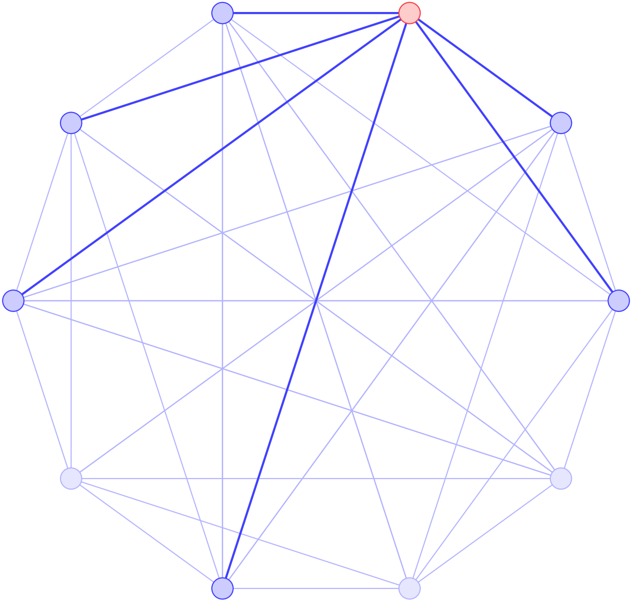

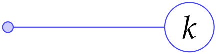
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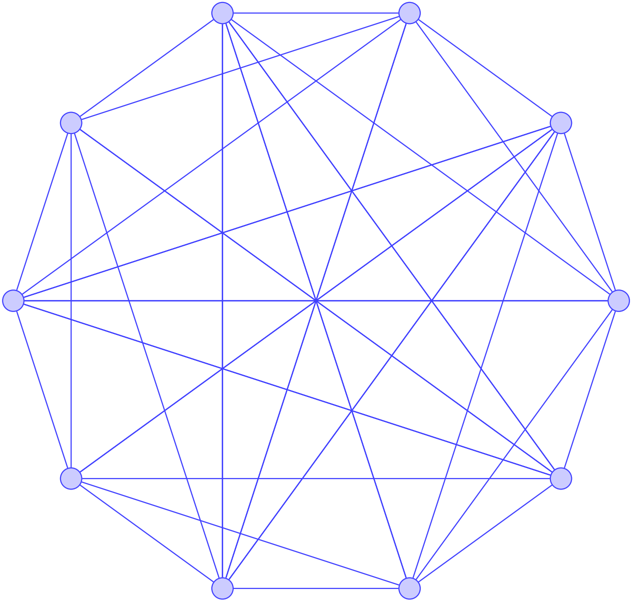
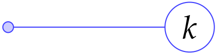
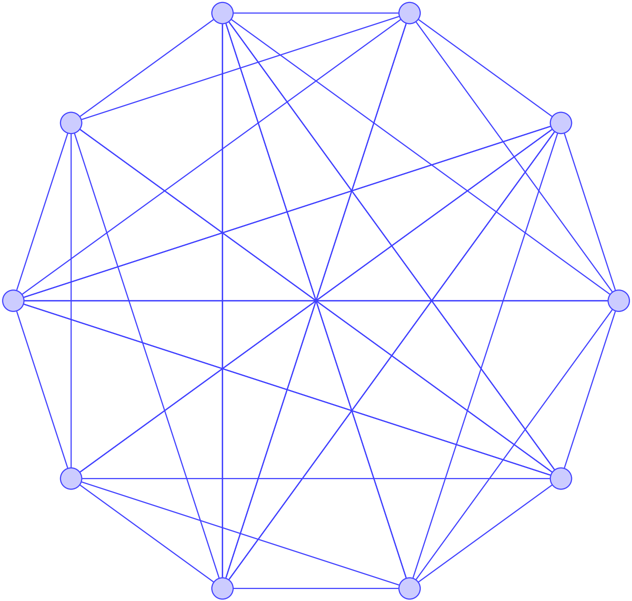
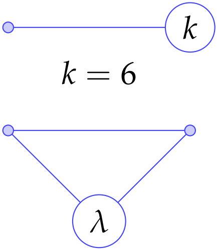




$k = 6$





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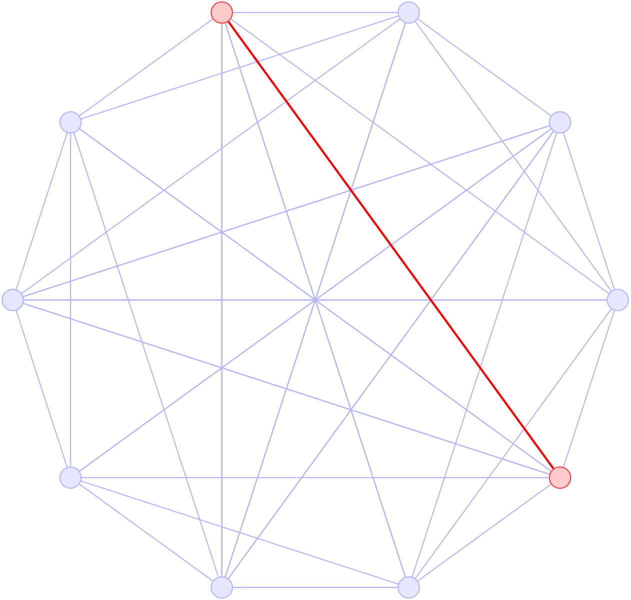
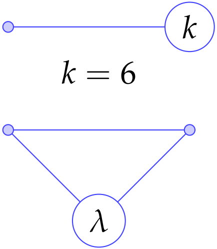
λ

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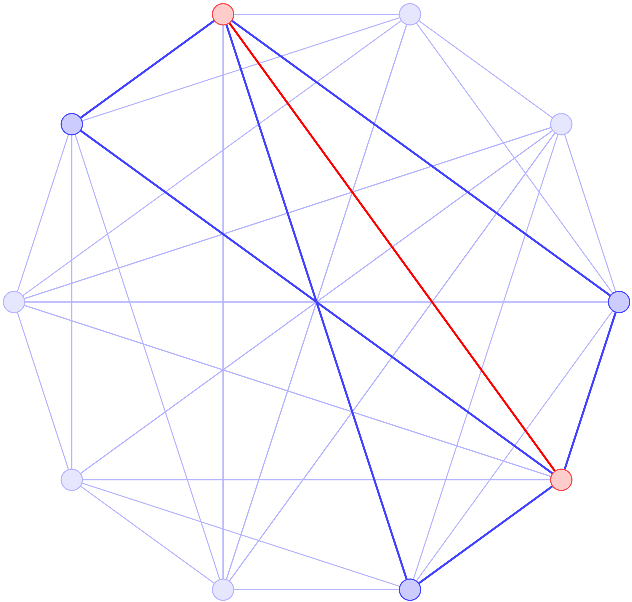

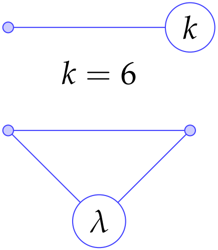
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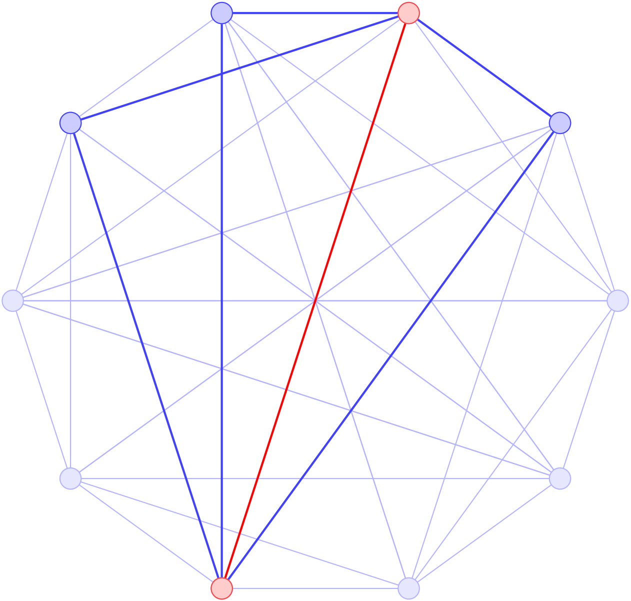

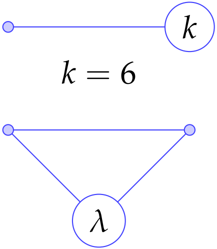
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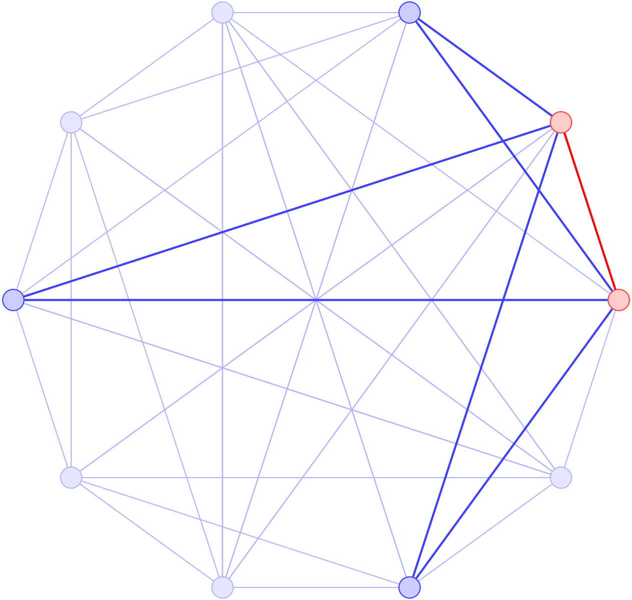

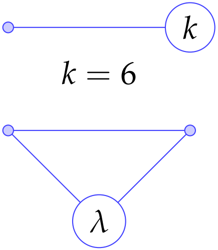
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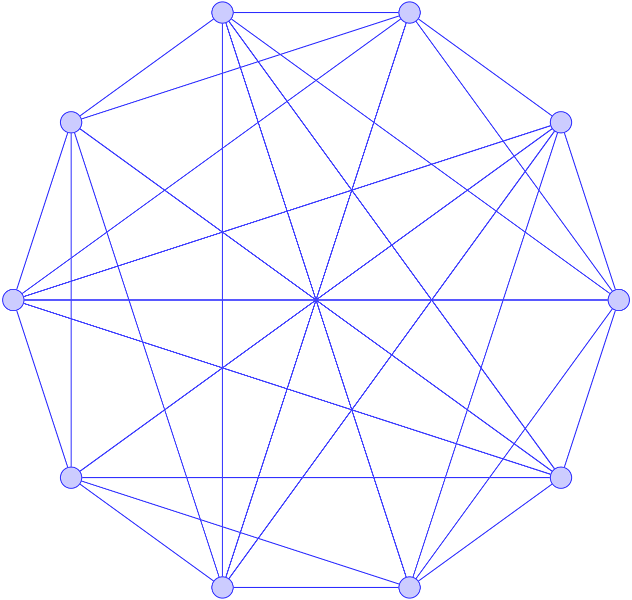

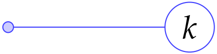

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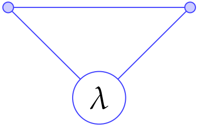
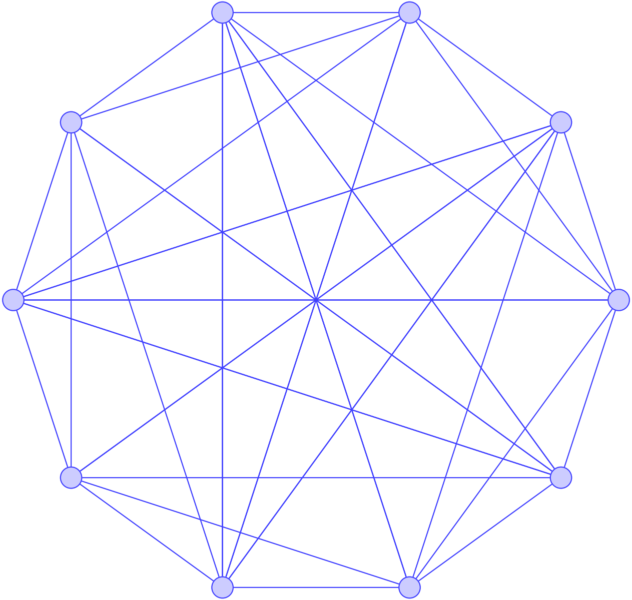
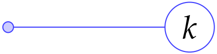


$$\lambda = 3$$





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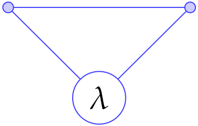

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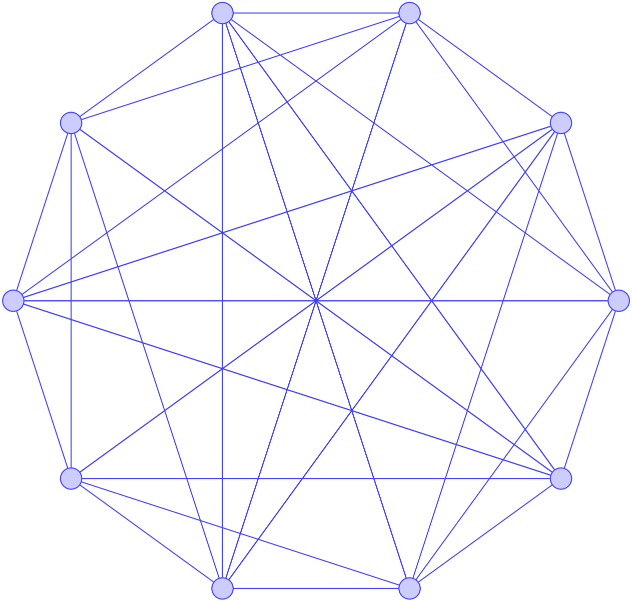
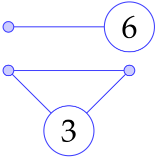
edge-regular

$\mathrm{erg}(10, 6, 3)$





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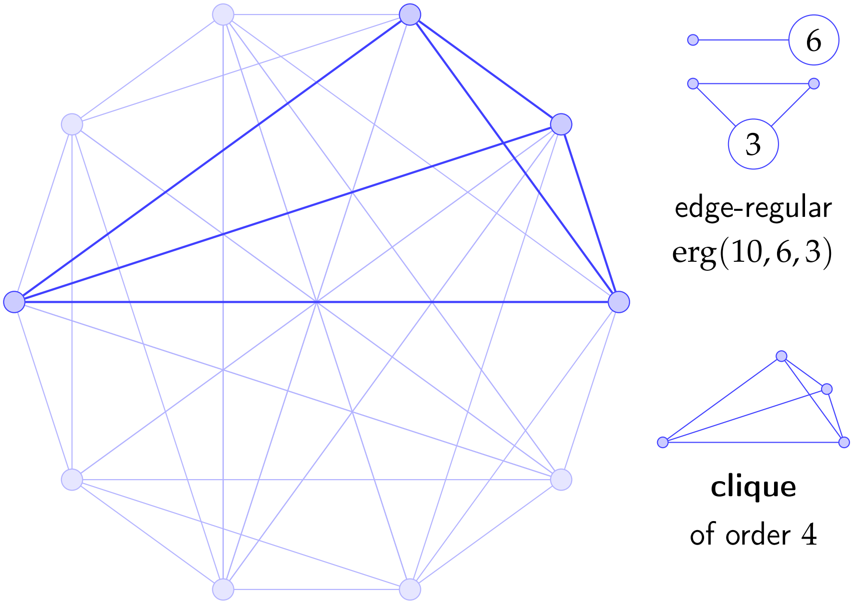
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edge-regular
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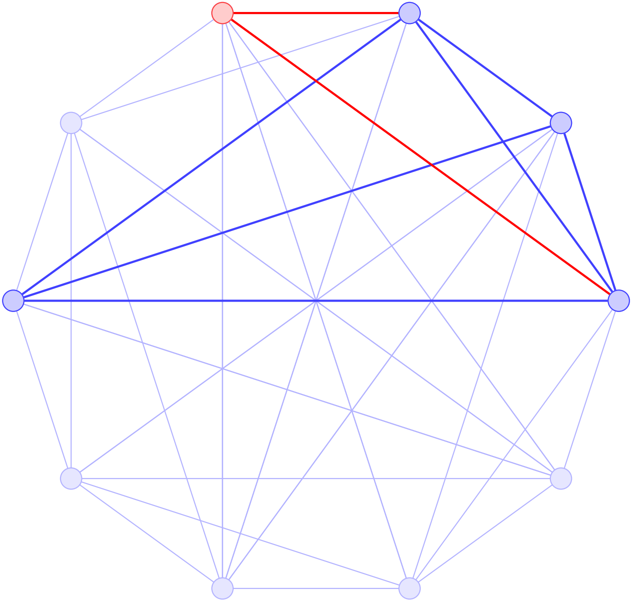
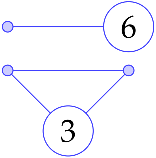
clique
of order 4





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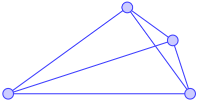

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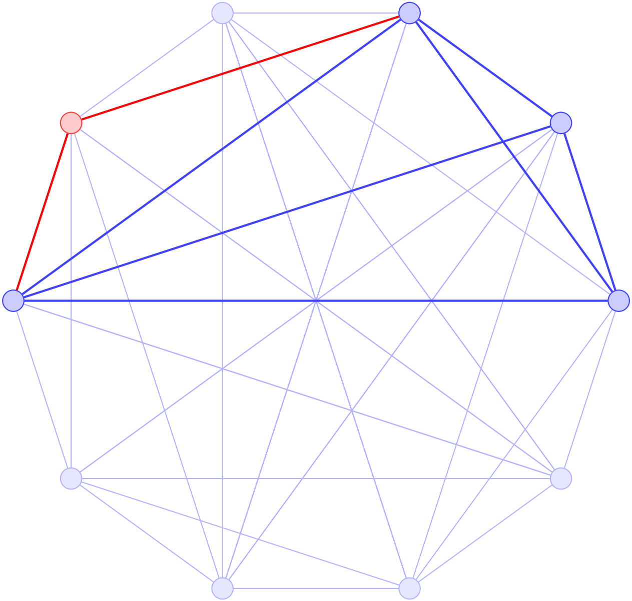

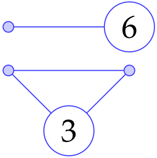

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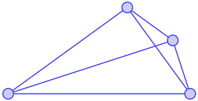

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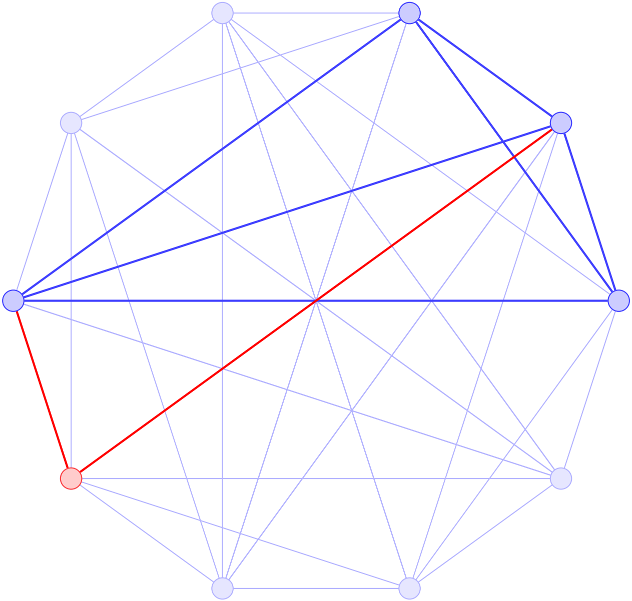

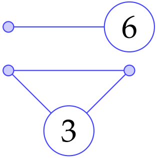

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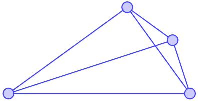

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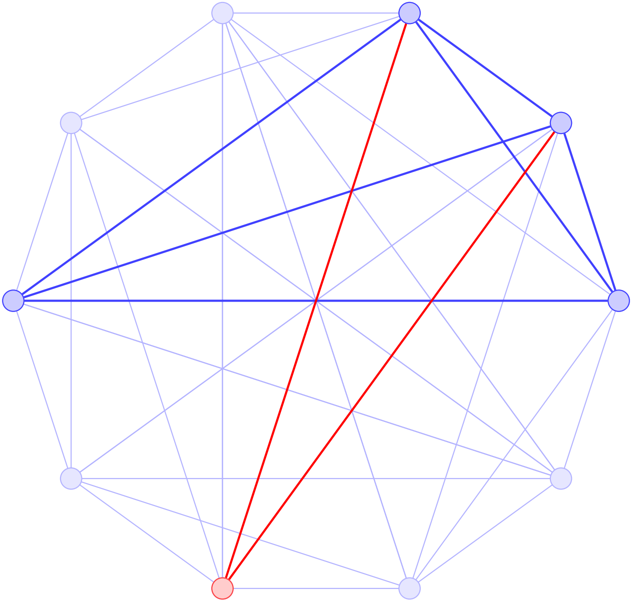

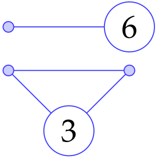

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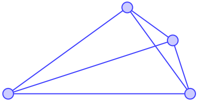

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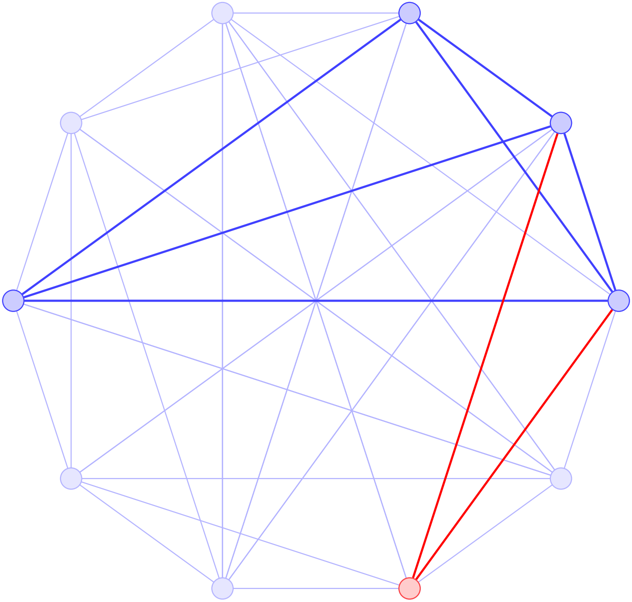

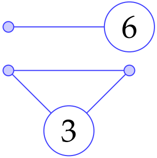

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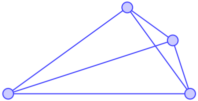

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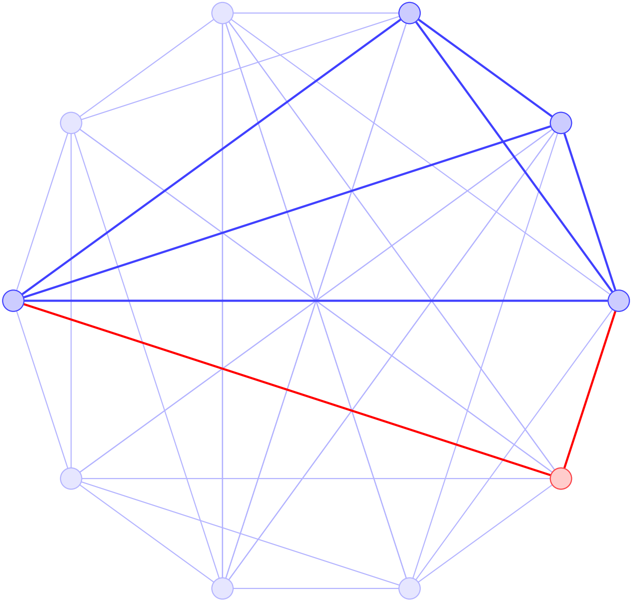

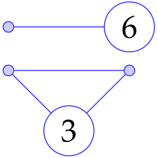

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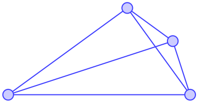

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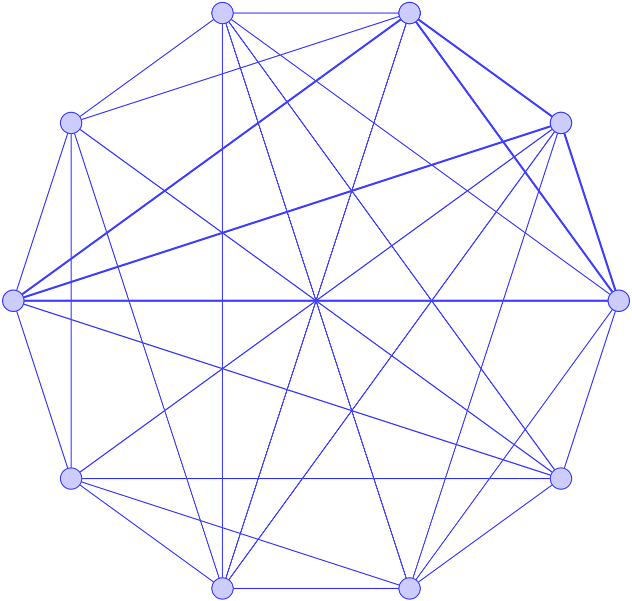

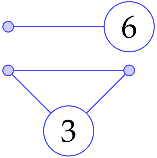

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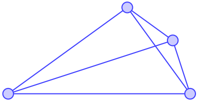

2-**regular** clique
of order 4

Theorem (Neumaier 1981)

Let Γ be edge-regular with a regular clique.
Suppose Γ is vertex-transitive and edge-transitive.
Then Γ is strongly regular.

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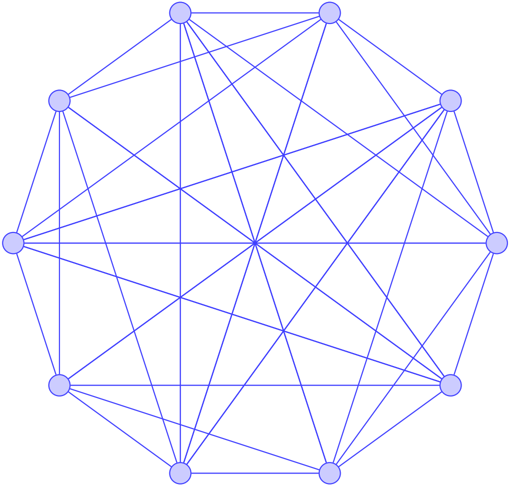

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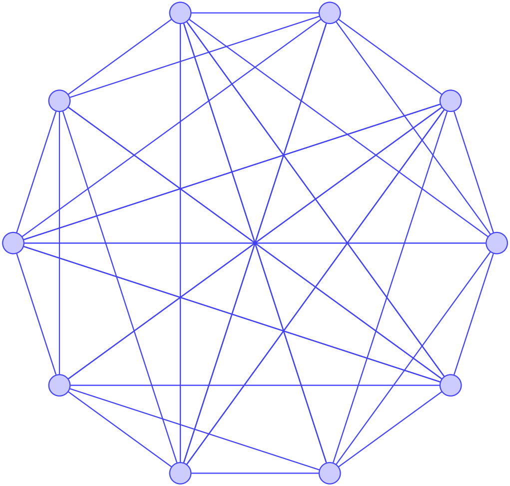

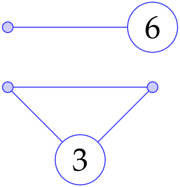

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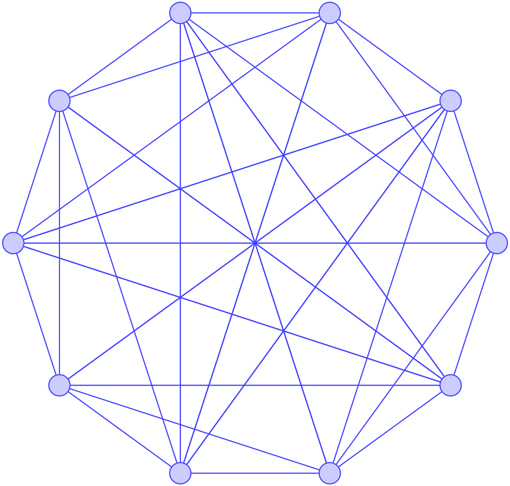

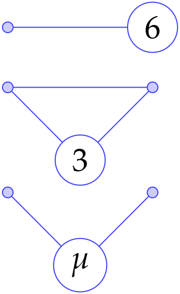

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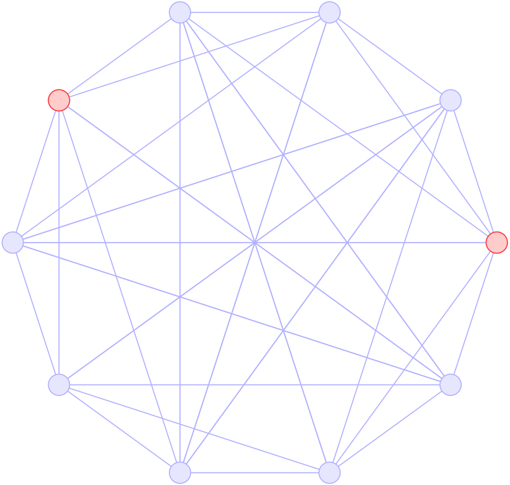

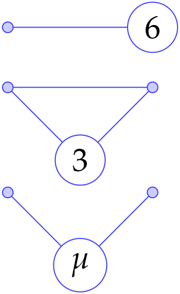

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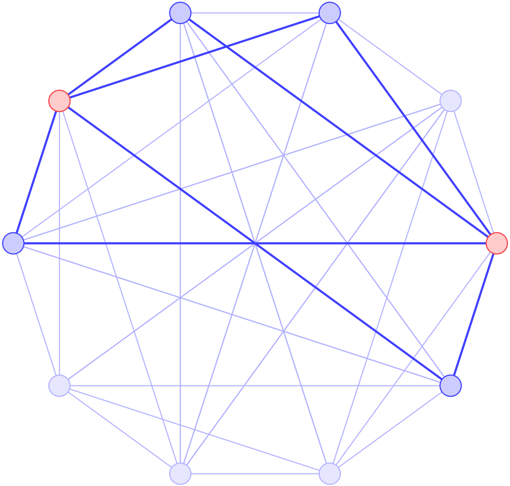

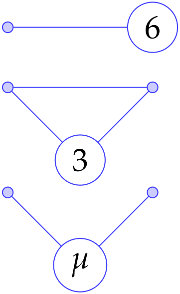

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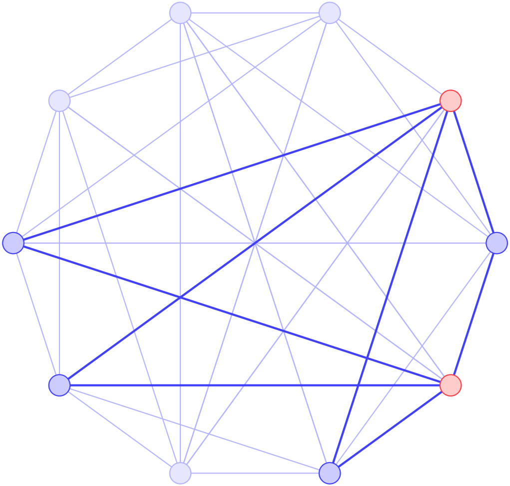

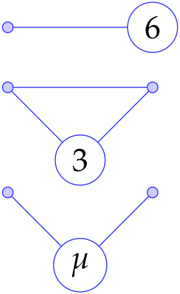

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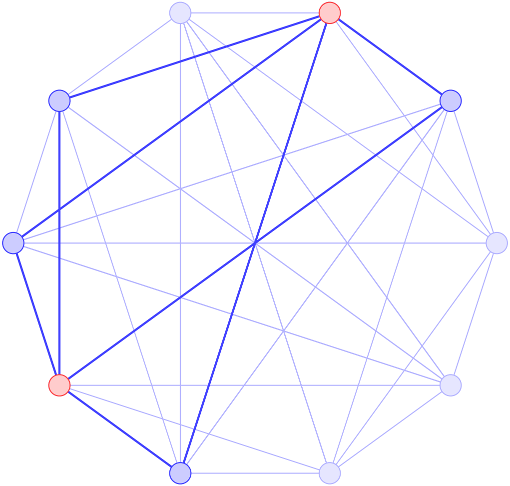

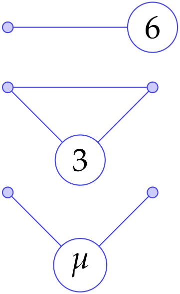

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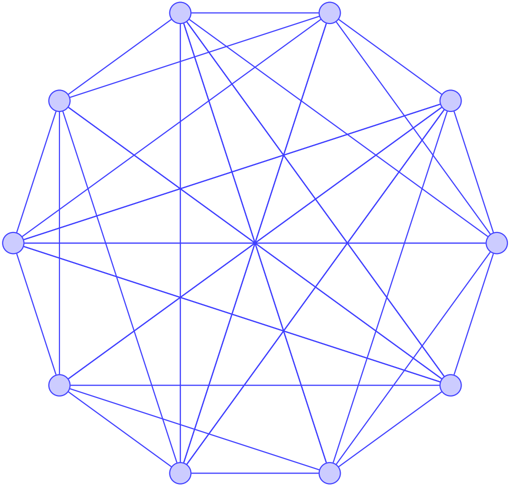

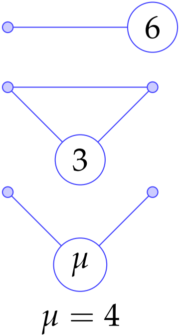

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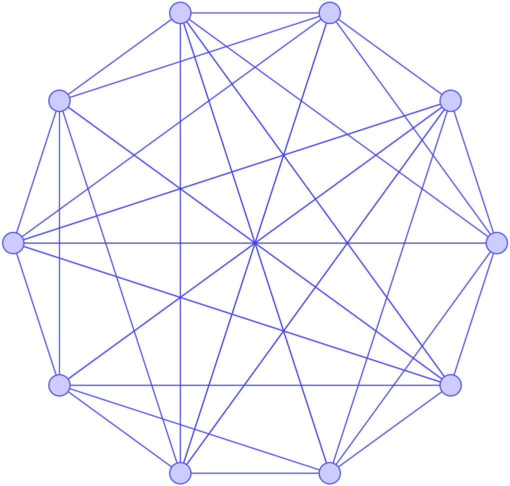

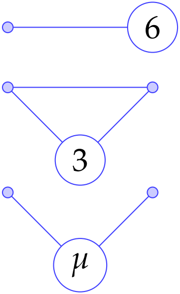

$\mu = 4$

strongly regular
$\mathrm{srg}(10, 6, 3, 4)$

Question (Neumaier 1981)

Is every edge-regular graph with a regular clique strongly regular?

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Answer (GG and Koolen 2017+)

No.

Question (Neumaier 1981)

Is every edge-regular graph with a regular clique strongly regular?

Answer (GG and Koolen 2017+)

No. There exist infinitely many non-strongly-regular, edge-regular vertex-transitive graphs with regular cliques.

Jack says "Hi"

Jack Koolen

Jack says "Hi"


Jack Koolen

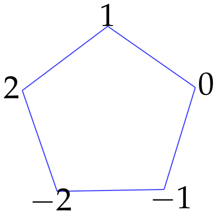
Cayley graphs

- Let G be an (additive) group and $S \subseteq G$ a (symmetric) generating subset, i.e., $s \in S \implies -s \in S$ and $G = \langle S \rangle$.

- The **Cayley graph** $\mathrm{Cay}(G, S)$ has vertex set G and edge set
$$\{\{g, g+s\} : g \in G \text{ and } s \in S\}.$$

Example

$\Gamma = \mathrm{Cay}(\mathbb{Z}_5, S)$ Generating set $S = \{-1, 1\}$



A construction

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$

Generating set S

$$\begin{array}{ll} (01,0) & (01,\pm 1) \\ (10,0) & (10,\pm 2) \\ (11,0) & (11,\pm 3) \end{array}$$

A construction

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$

- Γ is **edge-regular** $(28, 9, 2)$:

Generating set S

$$\begin{array}{cc} (01,0) & (01,\pm 1) \\ (10,0) & (10,\pm 2) \\ (11,0) & (11,\pm 3) \end{array}$$

A construction

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is **edge-regular** $(28, 9, 2)$:

Generating set S

$$\boxed{\begin{array}{cc} (01,0) & (01,\pm 1) \\ (10,0) & (10,\pm 2) \\ (11,0) & (11,\pm 3) \end{array}}$$

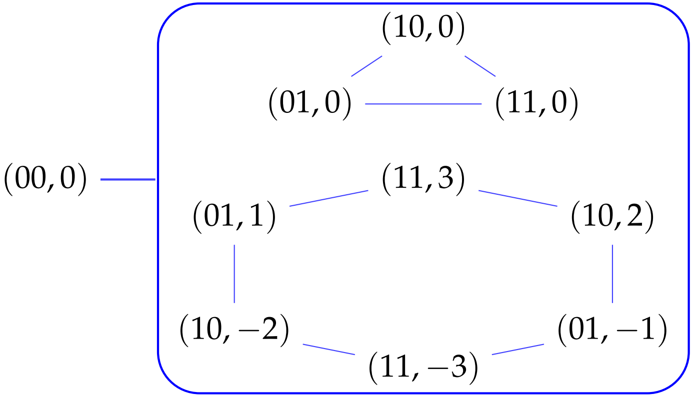
$(00, 0)$

A construction

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is **edge-regular** $(28, 9, 2)$:

Generating set S

$(01, 0)$	$(01, \pm 1)$
$(10, 0)$	$(10, \pm 2)$
$(11, 0)$	$(11, \pm 3)$



A construction

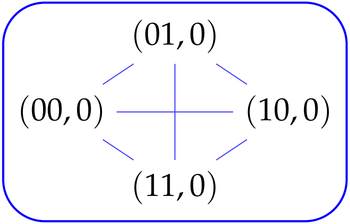
- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is edge-regular $(28, 9, 2)$;
- Γ has a 1-**regular** 4-**clique**:

Generating set S

$$\begin{array}{cc} (01, 0) & (01, \pm 1) \\ (10, 0) & (10, \pm 2) \\ (11, 0) & (11, \pm 3) \end{array}$$

A construction

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is edge-regular $(28, 9, 2)$;
- Γ has a 1-**regular** 4-**clique**:



Generating set S

$(01, 0)$	$(01, \pm 1)$
$(10, 0)$	$(10, \pm 2)$
$(11, 0)$	$(11, \pm 3)$

A construction

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is edge-regular $(28, 9, 2)$;
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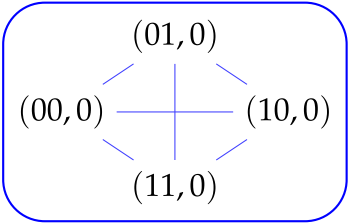
Generating set S

$$\begin{array}{cc} (01,0) & (01,\pm 1) \\ (10,0) & (10,\pm 2) \\ (11,0) & (11,\pm 3) \end{array}$$



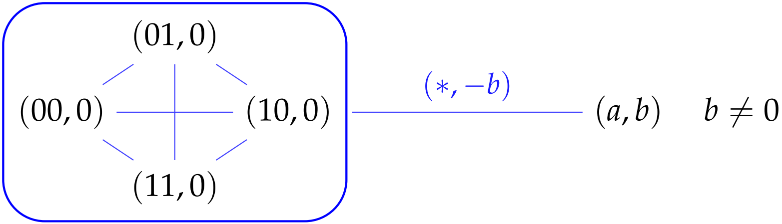
$(a,b) \quad b \neq 0$

A construction

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is edge-regular $(28, 9, 2)$;
- Γ has a 1-**regular** 4-**clique**:

Generating set S

$(01, 0)$	$(01, \pm 1)$
$(10, 0)$	$(10, \pm 2)$
$(11, 0)$	$(11, \pm 3)$



A construction

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is edge-regular $(28, 9, 2)$;
- Γ has a 1-regular clique of order 4;

Generating set S

$$\begin{array}{cc} (01,0) & (01,\pm 1) \\ (10,0) & (10,\pm 2) \\ (11,0) & (11,\pm 3) \end{array}$$

- Γ is **not strongly regular**:

A construction

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is edge-regular $(28, 9, 2)$;
- Γ has a 1-regular clique of order 4;

Generating set S

$$\begin{array}{cc} (01,0) & (01,\pm 1) \\ (10,0) & (10,\pm 2) \\ (11,0) & (11,\pm 3) \end{array}$$

- Γ is **not strongly regular**:

$(00,0)$

$$\begin{array}{cc} & (11,0) \\ (01,0) & (01,1) \\ (11,3) & (10,0) \\ (10,-2) & (01,-1) \\ (11,-3) & (10,2) \end{array}$$

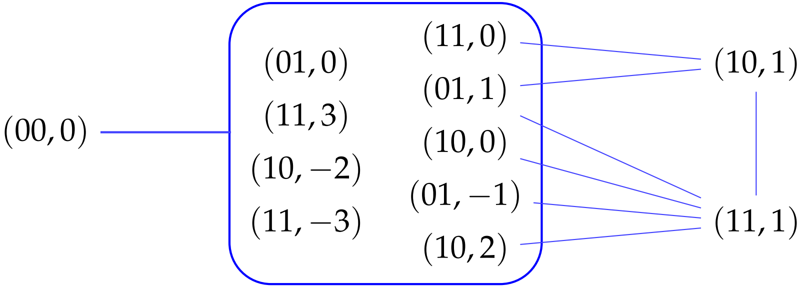
A construction

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is edge-regular $(28, 9, 2)$;
- Γ has a 1-regular clique of order 4;

Generating set S

$$\begin{matrix} (01,0) & (01,\pm 1) \\ (10,0) & (10,\pm 2) \\ (11,0) & (11,\pm 3) \end{matrix}$$

- Γ is **not strongly regular**:



A construction

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Generating set S

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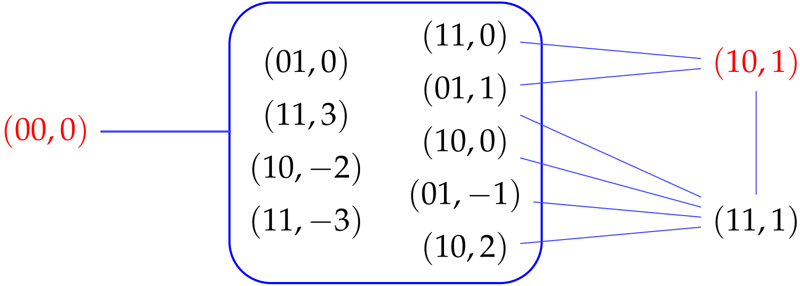

A construction

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Generating set S

$(01, 0)$	$(01, \pm 1)$
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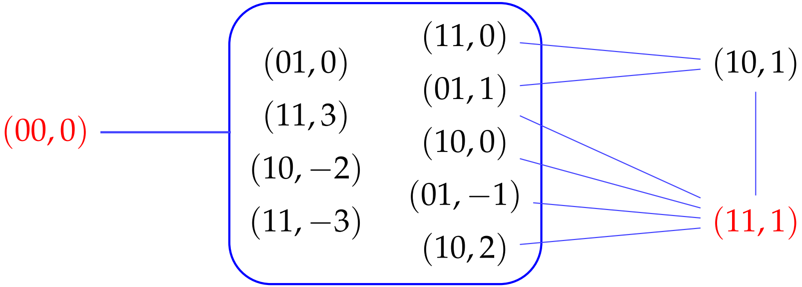

Some infinite families

- Generalise: $\mathbb{Z}_2^2 \oplus \mathbb{Z}_7$ to $\mathbb{Z}_{(c+1)/2} \oplus \mathbb{Z}_2^2 \oplus \mathbb{F}_q$.
- Works for $q \equiv 1 \pmod 6$ such that the 3rd cyclotomic number $c = c_q^3(1,2)$ is odd.
- Then there exists an $\mathrm{erg}(2(c+1)q, 2c+q, 2c)$ having a 1-regular clique of order $2c+2$.
- Take $p \equiv 1 \pmod 3$ a prime s.t. $2 \not\equiv x^3 \pmod p$. Then there exist a such that $c_{p^a}^3(1,2)$ is odd.

An example in the wild

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http://semr.math.nsc.ru

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MSC 05C

КЭЛИ-ДЕЗА ГРАФЫ, ИМЕЮЩИЕ МЕНЕЕ 60 ВЕРШИН

С.В. ГОРЯИНОВ, Л.В. ШАЛАГИНОВ

Abstract. Deza graph, which is the Cayley graph is called the Cayley-Deza graph. The paper describes all non-isomorphic Cayley-Deza graphs of diameter 2 having less than 60 vertices.

Keywords: Deza graph, Cayley graph, graph isomorphism, automorphism group.

1. Введение

В этой статье мы начинаем изучение графов Деза, которые являются графами Кэли. Графы Деза принято рассматривать как обобщение сильно регулярных графов. В ряде исследований было выяснено, что графы Деза наследуют некоторые свойства сильно регулярных графов. Например, в [1] показано, что вершинная связность графа Деза, полученного из сильно регулярного графа с помощью инволюции, совпадает с валетностью.

$\mathrm{erg}(24, 8, 2)$ with a 1-regular clique

Open problems

- Smallest non-strongly-regular, edge-regular graph with regular clique
- All known examples have 1-regular cliques
- Find general construction that includes $\mathrm{erg}(24, 8, 2)$

Willem Haemers,
Felix Lazebnik,
and Andrew Woldar;
Congratulations!