Characterizing identifying codes in a digraph or graph from its spectrum

Camino Balbuena, Cristina Dalfó, Berenice Martínez-Barona
Universitat Politècnica de Catalunya (UPC)

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1. Introduction
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Definition
Given a vertex subset $U \subset V$, let $N^-[U] = \bigcup_{u \in U} N^-[u]$. For a given integer $\ell \geq 1$, a vertex subset $C \subset V$ is a $(1, \leq \ell)$-identifying code in a digraph $D$ when, for all distinct subsets $X, Y \subset V$, with $1 \leq |X|, |Y| \leq \ell$, we get

$$N^-[X] \cap C \neq N^-[Y] \cap C.$$
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$$N^- [X] \cap C \neq N^- [Y] \cap C.$$ 

Remark
A digraph $D = (V, A)$ admits some $(1, \leq \ell)$-identifying code if and only if for all distinct subsets $X, Y \subset V$ with $|X|, |Y| \leq \ell$, we have

$$N^- [X] \neq N^- [Y].$$
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**Remark**
A digraph $D$ admits a $(1, \leq 1)$-identifying code if and only if $D$ is twin-free.
2. Non-spectral results for digraphs
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**Theorem**

Every 1-in-regular digraph $D$ admits a $(1, \leq 2)$-identifying code if and only if the girth of $D$ is at least 5.
2. Non-spectral results for digraphs

Theorem (Balbuena, D., Martínez-Barona, 2017)

Let $D$ be a 2-in-regular digraph. Then,

(i) $D$ admits a $(1, \leq 1)$-identifying code if and only if it does not contain any subdigraph isomorphic to the digraph of Figure 1 (a).

![Figure 1](image-url)
2. Non-spectral results for digraphs

Theorem (Balbuena, D., Martínez-Barona, 2017)

Let $D$ be a 2-in-regular digraph. Then,

$(ii)$ $D$ admits a $(1, \leq 2)$-identifying code if and only if it does not contain any subdigraph isomorphic to any of the digraphs of Figure 1.

Figure 1
2. Non-spectral results for digraphs

Corollary

Every oriented and $TT_3$-free 2-in-regular digraph admits a $(1, \leq 2)$-identifying code if and only if it does not contain any subdigraph isomorphic to Figure 1 (i).

Figure 1
Example

Circulant digraph $C(6; 1, 2)$. $X = \{0, 3\}$ and $Y = \{1, 4\}$, $N^-[X] = N^-[Y] = \{0, 1, 2, 3, 4, 5\} = V$, $C(6; 1, 2)$ cannot admit a $(1, \leq 2)$-identifying code. Subdigraph of Figure 1 $(\ell) \subset C(6; 1, 2)$.
2. Non-spectral results for digraphs

Example

Circulant digraph $C(13; 1, 4)$. $C'(13; 1, 4)$ is oriented, $TT_3$-free.

By the Corollary, $C(13; 1, 4)$ admits a $(1, \leq 2)$-identifying code. Since $N^-[X] \neq N^-[Y]$ for every two distinct sets $X, Y$ such that $2 \leq |X|, |Y| \leq 3$, $C(13; 1, 4)$ admits a $(1, \leq 3)$-identifying code.
Theorem (Balbuena, D., Martínez-Barona, 2017)

Let $D$ be a 2-in-regular digraph. $D$ admits a $(1, \leq 3)$-identifying code if and only if it is oriented, $TT_3$-free and does not contain any subdigraph isomorphic to any of the digraphs of Figure 2.

Figure 2
2. Non-spectral results for digraphs

Lemma
Let $D$ be a \textit{d-in-regular} digraph on $n$ vertices, without any of the subdigraphs of Figure 3. If $D$ admits a $(1, \leq \ell)$-identifying code, then $\ell \in \{d, d + 1\}$.

Figure 3
3. Spectral results for digraphs
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Lemma
Let \( D \) be a digraph with adjacency matrix \( A \) and with a set of eigenvalues denoted by \( \text{ev}(A) \). If \( -1 \notin \text{ev}(A) \), then \( D \) admits a \((1, \leq 1)\)-identifying code.

Remark
The converse is not true.

Example
The digraph of the figure has \( -1 \) as an eigenvalue, but it does admit a \((1, \leq 1)\)-identifying code.
3. Spectral results for digraphs

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The digraph of the figure has $-1$ as an eigenvalue, but it does admit a $(1, \leq 1)$-identifying code.
3. Spectral results for digraphs

Lemma
Let $D'$ be a digraph with maximum in-degree $\Delta^-$ having an eigenvalue $\lambda$ with eigenvector $x' = (x'_u)$, such that $x'_v = 0$ for any vertex $v \in V(D')$ with $d^-(v) < \Delta^-$. Then, any $\Delta^-$-in-regular digraph $D$ containing $D'$ as a subdigraph has also the eigenvalue $\lambda$. 
3. Spectral results for digraphs

Theorem

Let $D$ be a 2-in-regular digraph with adjacency matrix $A$.

(i) If $-1 \not\in \text{ev}(A)$ and $D$ does not contain any subdigraph isomorphic to (b), (c), (d), (f) and (i) of Figure 1, then $D$ admits a $(1, \leq 2)$-identifying code.

Figure 1
3. Spectral results for digraphs

Theorem
Let $D$ be a 2-in-regular digraph with adjacency matrix $A$.

(ii) If $-1, 0 \not\in \text{ev}(A)$ and $D$ does not contain any subdigraph isomorphic to (b)-(l) of Figure 2, then $D$ admits a $(1, \leq 3)$-identifying code.
3. Spectral results for digraphs

Proposition

Let $D = (V,E)$ be a digraph with adjacency matrix $A$ having some real eigenvalue, say $\lambda \in \text{ev}(A)$, different from the spectral radius. Let $x = (x_u)_{u \in V}$ be an eigenvector of $A$ associated to $\lambda$ such that $X = \mathcal{P}(x) = \{i : x_i > 0\}$ and $Y = \mathcal{N}(x) = \{i : x_i < 0\}$.

(a) If $\lambda < 0$, then $X \cup N^-(X) = Y \cup N^-(Y)$ \quad (\Leftrightarrow N^-[X] = N^-[Y]).$

(b) If $\lambda > 0$, then $X \cup N^-(Y) = Y \cup N^-(X)$.

(c) If $\lambda = 0$, then $N^-(X) = N^-(Y)$. 
3. Spectral results for digraphs

Corollary

Let $D$ be a digraph admitting a $(1, \leq \ell)$-identifying code. Let $A$ be its adjacency matrix having at least one negative eigenvalue $-\lambda$ (for $\lambda > 0$) with $x = (x_1, \ldots, x_n)$ any associated eigenvector. Then

$$\ell < \min_x \max \{ |\mathcal{P}(x)|, |\mathcal{N}(x)| \}.$$
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$$\ell < \min_x \max \{|P(x)|, |N(x)|\}.$$ 

Example

$\text{sp}(A) = \{0^4, 1^1, -1^1\}$.

Eigenvector corresponding to $-1$: $(0, -1, 1, -1, 0, 1)$.

$X = \{2, 5\}$, and $Y = \{1, 3\}$.

$N^{-}[X] = N^{-}[Y] = \{0, 1, 2, 3, 4, 5\}$.

It does not admit a $(1, \leq 2)$-identifying code.
3. Spectral results for digraphs

Lemma
Let $D$ be a digraph on $n$ vertices with adjacency matrix $A$, and let $\lambda$ be a real eigenvalue of $A$ with geometric multiplicity $m$. For any given index set $I \subset \{1, 2, \ldots, n\}$ with $|I| = m - 1$, there exists an eigenvector $x$ with eigenvalue $\lambda$ and entries $x_i = 0$ for every $i \in I$. 
4. The case of graphs
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Remark
The last Corollary can also be applied to graphs, which always have real eigenvalues.
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Theorem (Laihonen, 2008)
Let \( k \geq 2 \) be an integer.

1. If a \( k \)-regular graph has girth \( g \geq 7 \), then it admits a \((1, \leq k)\)-identifying code.

2. If a \( k \)-regular graph has girth \( g \geq 5 \), then it admits a \((1, \leq k - 1)\)-identifying code.
Example

Heawood graph: \( \text{sp}(H) = \{3, \sqrt{2}^6, -\sqrt{2}^6, -3\}\).

Eigenvectors:

\[ \begin{align*}
\lambda = -3 : & \quad (-1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1); \\
\lambda = -\sqrt{2} : & \quad (-1, 0, 1, 0, -1, \sqrt{2}, 0, -\sqrt{2}, 1, 0, 0, 0, 0), \\
& \quad (-1, 0, 1, -\sqrt{2}, 0, \sqrt{2}, -1, 0, 0, 0, 0, 1, 0), \\
& \quad (0, -1, \sqrt{2}, -1, 0, 1, -\sqrt{2}, 0, 0, 0, 1, 0, 0), \\
& \quad (0, -1, \sqrt{2}, 0, -\sqrt{2}, 1, 0, -1, 0, 1, 0, 0, 0), \\
& \quad (0, -\sqrt{2}, 1, 0, -1, \sqrt{2}, -1, 0, 0, 1, 0, 0, 0), \\
& \quad (-\sqrt{2}, 0, \sqrt{2}, -1, 0, 1, 0, -1, 0, 0, 0, 0, 1) \\
\end{align*} \]

With the last one: \( X = \{2, 5, 13\}, Y = \{0, 3, 7\} \).

\( N^-[X] = N^-[Y] = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 12, 13\} \).

It does not admit a \((1, \leq 3)\)-identifying code.

Laihonen’s Theorem: It is 3-regular and it has girth 6, it admits a \((1, \leq 2)\)-identifying code.
Thank you for your attention.

Gràcies per la vostra atenció.