Hyperspaces, compactifications, and N-linked systems
Roger W. Wardle
1982

The Vietoris topology $F$ on the hyperspace of non-empty closed subsets of a space $X$ is the supremum of the upper semi-finite ($U$) and lower semi-finite ($L$) topologies on the hyperspace. This hyperspace with the $U$ topology is denoted $U(X)$ and the hyperspace of $U(X)$ with the $L$ topology is denoted $L(U(X))$. The space $L(U(X))$ is at $T_{(0)}$ space which contains as subsets many different extensions of $X$. In particular, if $X$ is Tychonoff, the Stone-Cech compactification $(\beta X)$ is one such subset; this relies on the $z$-ultrafilter description of $(\beta X)$. As a subset of $L(U(X))$, $(\beta X)$ is compact but not closed. This fact allows us to represent each Hausdorff compactification of $X$ as the union of $X$ (L-HOOK) $(\beta X)$ and certain points of the boundary of $(\beta X)$. More generally, by altering the $U$ topology, we can do the same for each Wallman compactification of $X$.

The second part of the thesis generalizes ultrafilters in the following way: we call a collection of closed subsets of a $T_{(4)}$ space $X$ an $n$-linked system if and only if every subcollection of (LESSTHEQ) $n$ sets has non-empty intersection in $X$. The set $(SUMM)(,n)(X)$ of all maximal $n$-linked systems is a subset of $L(U(X))$; it is an extension of $X$ in which $X$ is not a dense subset. The space $(SUMM)(,2)(X)$ was discovered by J. de Groot and is called the superextension of $X$. It is a supercompact Hausdorff space, and has several nice properties. For $n$ (GREATERTHEQ) 3, $(SUMM)(,n)(X)$ has similar, but not so well-behaved, properties. It is not compact, but is cocompact, hence a Baire space. If $X$ is compact metric, then $(SUMM)(,n)(X)$ is metrizable, and conversely. For $n = 2$, $(SUMM)(,2)(X)$ is connected and locally connected if and only if $X$ is connected. Connectivity for $n$ (GREATERTHEQ) 3 is more difficult to prove. For example, we have $(SUMM)(,n)(X)$ connected if $X$ is a continuum which satisfies any one of the properties: (1) no finite subset separates $X$, (2) $X$ is linearly ordered, (3) $X$ is arcwise connected and metric. For each $j$ (ELEM) $\{2,...,n-1\}$ there is a topology $L(,j)$ for $(SUMM)(,n)(X)$ which is weaker than the subspace topology $L$; there is also a set $(SUMM)(,n)(,j)(X)$ derived from $(SUMM)(,n)(X)$ such that $((SUMM)(,n)(,j)(X),L)$ is a continuous open image of $((SUMM)(,n)(X),L(,j-1))$ (we take $L(,1) = L$).

Finally, we examine the $U$, $L$, and $F$ topologies for the hyperspaces of an arbitrary space $X$. Some very weak separation properties for $X$ characterize certain properties of the closure operator in $U(X)$ ($L(X),F(X)$) and determine when each space contains a homeomorphic copy of $X$. 