Perturbation Methods in Mathematical Finance: Zero-Coupon Bonds and Equivalent Volatilities
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2004

This thesis is concerned with two different problems from the area of Mathematical Finance: Zero-coupon bonds and European bond options. Zero-coupon bonds and bond options are two basic fixed-income financial instruments, and aside from being important subjects on their own right, they can be used as building blocks for pricing and managing risks of other vanilla instruments and some exotics. Among them: Swaps, caps, floors, swaptions, digitals, and Bermudans.

There are many different models and approaches in the literature to the problem of pricing zero-coupon bonds by one-factor models. A common feature is that exact closed-form formulas are always sought. The approach in this thesis is different in that exact is traded for approximate. Whenever exact formulas are not available, in general numerical methods are prescribed. Exclusively using them can detach the user from the nature of the problem. It is in this frame of mind that, in the first part of this work, approximate closed-form formulas for zero-coupon bonds under the generalized Black Karasinski (gBK) model are derived. The gBK model includes as particular cases the well-known CIR, BK, and HW models. The approximate formulas involve integrals which can be evaluated explicitly for some simple cases, which are rich enough to, at least, fit a given discount curve.

The context of the second part of this thesis is related to the common practice of using implied volatilities for quoting, pricing and hedging European options. The second part of this thesis considers yield models for bond options. Bond options are quoted in terms of price volatilities, not rate volatilities. Using yield-based models would allow us to directly compare them to swaptions, which are rate-based. Another reason favoring yield-based models would be the fact that (interest) rates are the actual random quantity, while (bond) prices move in response. Approximate formulas for *Implied Volatilities*, called *Equivalent Volatilities*, are derived under one and two-factor models.

Regular and Singular Perturbation Methods are used to derive the approximate formulas. Alternative solution methods traditionally found in the literature are finite-difference-discretizations of the governing partial differential equations and Monte Carlo simulations. Perturbation Methods are superior to these solution methods, whenever they can be applied, of course, in two different aspects: By the computational speed, specially when calibration is involved, and by giving valuable insight into the nature of the problem.

From the mathematical point of view the first and second parts can be classified as regular and singular perturbation problems, respectively. The first part involves solving the direct problem of pricing zero-coupon bonds. The second part involves solving the direct problem of option pricing and subsequently a related inverse problem which solves for one of the parameters in the model: The volatility.