The Analysis and Numerical Simulation of a Mathematical Model of Bone Growth and Reabsorption.
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This thesis concerns the extension of a mechanical model of bone remodeling from its original form to include the mathematical analysis of two of its constituent geometric forms, the rod and the plate. The original mechanical model included a force balance equation, an equation for the strain tensor, an equation for the stress tensor either in terms of the strain tensor or in terms of the strain tensor and its time derivative, equations for surface forces as well as a remodeling rate equation. This model was augmented with the use of the analysis of the adaptive elastic rod used to model trabecular bone. The equation for the stress tensor was recast in light of the asymptotically thin, elastic rod which gave rise to asymptotic expansions for the displacement vector, the stress tensor, the strain tensor as well as for the change in a reference value of one pertaining to the original mass of solid bone found in the elastic rod.

Medical imaging techniques have shown that trabecular bone is more realistically approximated by a combination of rods and plates. Thus, the above analysis was undertaken again for the elastic plate. Furthermore, dissipation of acoustic energy in soft tissue interrogation by ultrasound indicates viscous properties for cancellous bone in the poroelastic bone matrix. Thus, Kelvin-Voigt viscoelastic cases for both the rod and the plate underwent the same analysis as in the elastic cases.

A combined, elastic rod-plate model was conceived for the purpose of numerical simulations. The leading order terms in the asymptotic expansions of the elastic rod and of the elastic plate were utilized for this end. First, displacement and bone growth were shown for the elastic rod and the elastic plate separately. Then, results dealing with the combined, elastic rod-plate model were included along with consideration of the SMI or Structure Model Index, a measure of the relative proportions of rods to plates in bone specimen. Numerical simulations of the displacement and growth of a simple isotropic viscoelastic rod were also completed with the results included herein.

Existence and uniqueness for the solution of the ODE that is the remodeling rate equation for a viscoelastic system is proven via Picard's method of successive approximations. A uniqueness argument for a viscoelastic system with general boundary conditions is also presented.