On some extremal properties of bipartite graphs of large girth
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This dissertation is concerned with some extremal properties of bipartite graphs of large girth. A near-linear space has a representation as a bipartite point-line incidence graph so it is natural to consider the extremal properties of such a graph. Every bipartite graph with bipartition classes of sizes $m$ and $n$, $(n \geq m)$ may be embedded in a bipartite graph with bipartition classes of equal cardinality $n$. Therefore, with the exception of one theorem in Chapter 2, we concentrate on the properties of bipartite graphs, where the bipartition has parts of equal size.

Define $G=G(n,n)$ to be a simple, bipartite graph of girth $g$ with partition classes of equal cardinality $n$ and $e$ edges. It has been shown in (Bol79) that for $g \geq 6$ the number of edges of $G$ is bounded above by $r \cdot n$ where $r$ is the positive root of the equation $x^2 - x + 1 - n = 0$. C. P. Teo and K. M. Koh proved in (TK92) that if $G$ is 2-connected the number of cycles of length 6 is bounded above by $[e/6](e - 2n+1)$. Equality holds in both instances if and only if $G$ is the point-line incidence graph of a finite projective plane of order $q=r - 1$.

We will provide a proof of a bound on the number of 6-cycles that depends only on $n$ using these two previous results that does not require that $G$ be 2-connected. A second proof of this bound is presented as a corollary to a theorem on a bound on the number of 6-cycles for the larger class of bipartite graphs with partition classes of cardinality $n$, $m$ and girth at least six. This second approach does not rely on the previous results.

Furthermore, for $g \geq 8$ we show under certain conditions that the number of edges of $G$ is bounded above by $r \cdot n$ where $r$ is the positive real root of the equation $x^3 - 2x^2 + 2x - n = 0$. As a consequence of this we also obtain an upper bound on the number of cycles of length eight in $G$ with equality in both cases if and only if $G$ is a generalized 4-gon (see definition immediately preceding Theorem 1.2). In addition, we draw some conclusions on a bound for the number of edges if these conditions do not hold. We also speculate on the existence of similar bounds for bipartite graphs of girth $g \geq 10$. 